

Weyl's Character Formula for Non-connected Lie Groups and Orbital Theory for Twisted Affine Lie Algebras

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We generalize I. Frenkel's orbital theory for non twisted affine Lie algebras to

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measure.

1. INTRODUCTION

In [4], Frenkel develops a generalization of Kirillov's orbit theory for finite dimensional Lie groups [10] in the case of untwisted affine Lie algebras. In particular, he classifies affine (co-) adjoint orbits of the underlying loop groups in terms of conjugacy classes of connected compact Lie groups. Using this classification and the theory of Wiener integration on compact Lie groups, Frenkel obtains an interpretation of the character of a highest weight representation as an integral over an associated orbit in the coadjoint representation.

The aim of this paper is to generalize Frenkel's orbital theory to the case of twisted affine Lie algebras. Kleinfeld ([11]) already made a first step towards the adaptation of Frenkel's theory to this more general case, but he was unable to interpret the character formula as an orbital integral. In order to obtain an interpretation of the formulas as orbital integrals we shall introduce certain non-connected compact Lie groups, so called principal extensions, into the geometrical picture. It turns out that the affine orbits of the adjoint representation of a twisted loop group are parametrized by the conjugacy classes in the "outer component" of such a group. With this result and a character formula for non-connected Lie groups at hand we are able to translate Frenkel's program to the twisted case. I now give a brief description of the contents of this paper.

Section 2 contains some facts about non-connected compact Lie groups, some of them well known. After constructing the principal extension \tilde{G} of a semisimple compact group G , we derive an analogue of the Weyl character formula for the connected components of \tilde{G} not containing the identity. The proof of this formula involves an analogue of the Weyl integral formula for G -invariant functions on \tilde{G} , and the character itself is governed by the dual of a certain “folded” root system. That is, if R is the root system of G and τ is a diagram automorphism of R then τ acts as an outer automorphism on G , and the characters on the connected components of \tilde{G} are governed by the root system R^{τ^\vee} , the dual to the “folded” root system R^τ of the fixed point group G^τ . It is worthwhile to note that no group belonging to the root system R^{τ^\vee} can, in general at least, be realized as a subgroup of G in contrast to G^τ . The group G acting on \tilde{G} by conjugation, we shall view each connected component of \tilde{G} as a G -manifold. As another direct application of the integral formula, we compute the radial component of the Laplacian on the connected components of \tilde{G} with respect to that G -action.

In Section 3, we study affine orbits of the adjoint representation of a twisted loop group $\mathcal{L}(G, \tau)$. By a slight alteration of Frenkel’s original methods we see that for G compact, every such orbit contains a constant loop and the orbits in certain affine “shells” are parametrized by the G -orbits in the connected component of \tilde{G} containing τ . To be more precise and using different terminology, Frenkel regards a loop into the Lie algebra \mathfrak{g} as a connection on a principal, trivial fibre bundle over the circle S^1 with structure group G and he associates to this loop the monodromy of this connection which is an element of G . The action of $\mathcal{L}(G)$ on an affine shell in the affine Lie algebra is then given by gauge transformations and it is compatible with the G -conjugation on the monodromies in G . That way he obtains a well defined bijection from the adjoint orbits of $\mathcal{L}(G)$ (of some fixed affine shell) to the set of conjugacy classes in G . In the case of a twisted loop group $\mathcal{L}(G, \tau)$ where τ is a diagram automorphism of G the corresponding map will neither be surjective nor injective (cf. [11]). In this case, it is appropriate to replace the monodromy with the “ $\frac{1}{r}$ monodromy” (i.e., “monodromy” after $\frac{1}{r}$ of the full circle) multiplied by τ . Here r is the order of τ . The gauge action of $\mathcal{L}(G, \tau)$ is then compatible with conjugation in the component $G\tau$, and we are able to classify the affine adjoint orbits of $\mathcal{L}(G, \tau)$ in terms of conjugacy classes in $G\tau$.

In Section 4, we use the theory of Wiener integration on a non-connected compact Lie group to rewrite the irreducible highest weight characters of a twisted affine Lie algebra as an integral over the space of paths inside the connected component of \tilde{G} containing the element τ . After a brief summary of some results about affine Lie algebras and their representations in Section 4.1, we shall show in Sections 4.2 and 4.3 how the irreducible

highest weight characters of a twisted affine Lie algebra are linked to the fundamental solution of the heat equation on a non-connected compact Lie group. At this point, the characters on the connected component $G\tau$ enter the picture. The fundamental solution of the heat equation is used in Section 4.4 to define the Wiener measure on the space of paths in the connected component of a compact Lie group. Computing a certain integral with respect to this measure, we can rewrite the affine characters as an integral over a path space. In Section 4.5 we then show, adopting the original procedure of [4], how this integral can be interpreted as an integral over a coadjoint orbit of the corresponding twisted loop group thus completing Frenkel's program for twisted affine algebras as well.

2. INTEGRATION AND CHARACTER FORMULAS FOR NON-CONNECTED LIE GROUPS

2.1. Principal Extensions and Conjugacy Classes

Let G be a simply connected semisimple compact Lie group, T a maximal torus in G , and R the root system of G with respect to T . If W is the Weyl group of R (and G) and Γ is the group of diagram automorphisms of the Dynkin diagram of G then we have $\text{Aut}(R) = W \rtimes \Gamma$. Every $\tau \in \Gamma$ can be lifted to an automorphism of G in the following way. Let Π be a basis of R . If \mathfrak{g} is the Lie algebra of G and $\mathfrak{g}_{\mathbb{C}}$ its complexification, we choose a set $\{e_{\alpha}, f_{\alpha}, h_{\alpha}\}_{\alpha \in \Pi}$ of Chevalley generators of $\mathfrak{g}_{\mathbb{C}}$ and set $\tau(e_{\alpha}) = e_{\tau(\alpha)}$. This extends to a Lie algebra automorphism of $\mathfrak{g}_{\mathbb{C}}$ which leaves \mathfrak{g} invariant. Since we chose G to be simply connected, τ can again be lifted to an automorphism of G leaving T invariant. Thus we have defined a homomorphism $\varphi: \Gamma \rightarrow \text{Aut}(G)$, and we can set $\tilde{G} = G \rtimes_{\varphi} \Gamma$, and call \tilde{G} the principal extension of G . Obviously we have $\tilde{G}/G = \Gamma$.

If G is not simply connected we have $G = \bar{G}/K$, where \bar{G} denotes the universal covering group of G , and K is some subgroup of the center of \bar{G} . Let Γ_K be the subgroup of Γ which leaves K fixed. Then the group $G \rtimes_{\varphi} \Gamma_K$ is called the principal extension of G . The principal extensions of compact Lie groups are, in general, central extensions of the automorphism groups of the compact groups and play a crucial role in the structure theory of the non-connected compact groups (cf. [17]).

G acts on the components of \tilde{G} by conjugation. The orbits of this action on the component $G\tau$, the connected component of \tilde{G} containing τ , are parametrized by a component of the space $S/W(S)$, where S is a Cartan subgroup of \tilde{G} in the sense of [2] such that $\tau \in S$. That is in our cases $S = T^{\tau} \times \langle \tau \rangle$ where T^{τ} is the connected component containing the identity of the τ -invariant part of the maximal torus T . The group $W(S) = N(S)/S$ is a finite group and is called the Weyl group belonging to S . In particular,

if $S_0 = T^\tau$ is the connected component of S containing e , then every element of $G\tau$ is conjugate under G to an element of $S_0\tau$, and two elements of $S_0\tau$ are conjugate under G if and only if they are conjugate in $N(S)$. Furthermore, S_0 is regular in G ; i.e., there is a unique τ -invariant maximal torus of G containing S_0 , cf. e.g. [2].

2.2. A "Weyl Integral Formula" for $G\tau$

Let G be a connected semisimple Lie group of type A_n , D_n , or E_6 and \tilde{G} its principal extension. Since the other connected Dynkin diagrams do not admit any diagram automorphisms, the principal extensions of the corresponding compact Lie groups are trivial. We want to derive an analogue of the Weyl integral formula for G -invariant functions on the component $G\tau$ of \tilde{G} . Let $S \subset \tilde{G}$ be a Cartan subgroup of G containing τ such that S/S_0 is generated by $S_0\tau$, and let (\cdot, \cdot) be the negative of the Killing form on \mathfrak{g} . Since G is semisimple, this gives an $\text{Ad}(G)$ invariant scalar product on \mathfrak{g} , and \mathfrak{g} decomposes into a direct sum $\mathfrak{g} = LT \oplus L(G/T)$, where LT is the Lie algebra of T and $L(G/T)$ its orthogonal complement. In the same way, LT decomposes into $LT = LS_0 \oplus L(T/S_0)$, hence we have $\mathfrak{g} = LS_0 \oplus L(T/S_0) \oplus L(G/T)$.

There are normalized left invariant volume forms $d(gS_0)$, ds , and dg on G/S_0 , S_0 and G which are unique up to sign (i.e., orientation). The form dg defines a volume form on $G\tau$ by right translation with τ , which will be called dg as well. Hence, on G , we have $|\Gamma_K| \cdot d\tilde{g} = dg$, where $d\tilde{g}$ is the normalized left invariant volume form on \tilde{G} .

The projection $\pi: G \rightarrow G/S_0$ induces a map $D\pi: \mathfrak{g} \rightarrow T_{eS_0}G/S_0$ which maps $L(G/S_0) := L(G/T) \oplus L(T/S_0)$ isomorphically to the tangent space of G/S_0 at the point eS_0 . Hence we can identify these spaces via $D\pi$. Let $n = \dim G$ and $k = \dim S_0$. Then $\pi^* d(gS_0)$ is a left invariant $(n-k)$ -form on G . Using the k -form $ds_e \in \text{Alt}^k(LS_0)$ we get $\text{pr}_2^* ds_e \in \text{Alt}^k \mathfrak{g}$ where $\text{pr}_2: \mathfrak{g} = L(G/S_0) \oplus LS_0 \rightarrow LS_0$ is the second projection. The form $\text{pr}_2^* ds_e$ defines a left-invariant k -form β on G by left translation, so $\pi^* d(gS_0) \wedge \beta$ is a volume form on G . Hence we have $\pi^* d(gS_0) \wedge \beta = cdg$. We may choose the signs so that $c > 0$, and it is not hard to see that in this case $c = 1$ (cf. [2]).

There is a volume form $\alpha = \text{pr}_1^* d(gS_0) \wedge \text{pr}_2^* ds$ on $G/S_0 \times S_0$. Identifying \mathfrak{g} with $L(G/S_0) \oplus LS_0$ and evaluating the forms at the unit element, one finds $\alpha_{(eS_0, e)} = dg_{(e)}$.

Since every element of $G\tau$ is conjugate under G to an element of $S_0\tau$, the map

$$\begin{aligned} q: G/S_0 \times S_0 &\rightarrow G\tau, \\ (gS_0, s) &\mapsto g s \tau g^{-1} \end{aligned}$$

is surjective, and by the above we get $q^* dg = \det(q) \alpha$.

LEMMA 2.1. *The functional determinant of the conjugation map q is given by*

$$\det(q)(gS_0, s) = \det(\text{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)}).$$

Proof. Similar to the proof of [2, Proposition IV, 1.8]. (For more details on the proofs in this section see [18].) ■

LEMMA 2.2. *Let z be a generator of S . Then*

- (i) $|q^{-1}(z)| = |W(S)|$.
- (ii) *For $(g_1S_0, s_1), (g_2S_0, s_2) \in q^{-1}(z)$ we have*

$$\det(q)(g_1S_0, s_1) = \det(q)(g_2S_0, s_2).$$

- (iii) *There exists a generator z of S such that q is regular in each $(gS_0, s) \in q^{-1}(z)$.*

Proof. Similar to the proof of [2, Proposition IV, 1.9]. ■

From this we obtain the mapping degree of q , $\deg(q) = \text{sign}(\det(q)) \cdot |W(S)|$. Thus using Fubini's theorem, one gets

PROPOSITION 2.3. *Let $f: G\tau \rightarrow \mathbb{R}$ be a G -invariant function. Then*

$$\int_{G\tau} f(g) dg = \frac{1}{|W(S)|} \cdot \int_{S_0} f(s\tau) \cdot |\det(\text{Ad}|_{L(S/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)})| ds.$$

In order to obtain an analogue of the classical Weyl integral formula, one has to calculate the functional determinant of q in terms of the root system of G . We adapt the notation used in [2]. That is, if $\alpha \in LT^*$ is an infinitesimal root of G , then ϑ_α denotes the corresponding global root $T \rightarrow S^1$. So for $H \in LT$ one has $\vartheta_\alpha \circ \exp(H) = e^{2\pi i \alpha(H)}$. Setting $e(x) = e^{2\pi i x}$, we get $\vartheta_\alpha = e(\alpha)$.

Now we consider the action of $(\text{Ad}(s\tau) - I)$ on $L(G/S_0)_\mathbb{C} = \mathbb{C} \otimes L(G/S_0)$. As before, $L(G/S_0)_\mathbb{C}$ decomposes into two orthogonal subspaces

$$L(G/S_0)_\mathbb{C} = L(T/S_0)_\mathbb{C} \oplus L(G/T)_\mathbb{C}.$$

Let $X \in L(T/S_0)$ be an eigenvector of $\text{Ad}(\tau)$. A short calculation shows

$$(\text{Ad}(s\tau) - I)(X) = -X + \gamma X,$$

where $\gamma = \pm 1$ if $\tau^2 = e$, and γ is a third root of unity if $\tau^3 = e$. We can choose (gS_0, s) to be a regular point of q , so $\gamma \neq 1$ and $(\text{Ad}(s\tau) - I)(X) = -2X$ if $t^2 = e$, and $(\text{Ad}(s\tau) - I)(X) = (\gamma - 1)X$ if $\tau^3 = e$.

Now $L(G/T)_{\mathbb{C}}$ decomposes into the direct sum of root spaces

$$L(G/T)_{\mathbb{C}} = \bigoplus_{\alpha \in R} L_{\alpha},$$

and for $X \in L_{\alpha}$, $s \in S_0$, one has

$$\text{Ad}(s)(X) = \vartheta_{\alpha}(s)(X).$$

Let $\bar{\alpha} = \alpha|_{LS_0}$ for $\alpha \in R$. It is a well known fact that the set $R^{\tau} = \{\bar{\alpha} \mid \alpha \in R\}$ is a (not necessarily reduced) root system. The relation between the type of R and the type of R^{τ} is shown in the following table:

R	A_{2n-1}	A_{2n}	$D_n(n \geq 4)$	D_4	E_6
$\text{ord}(\tau)$	2	2	2	3	2
R^{τ}	C_n	BC_n	B_{n-1}	G_2	F_4

So if R is of type A_{2n-1} , D_n , or E_6 , then R^{τ} is a reduced root system. It is easy to see that in this case, $\bar{\alpha}$ is a long root of R^{τ} if and only if $\tau(\alpha) = \alpha$. Otherwise $\bar{\alpha}$ is a short root of R^{τ} . If R is of type A_{2n} , then the root system R^{τ} is not reduced and three distinct root lengths occur. In this case, $\bar{\alpha}$ is a long root in R^{τ} if α is invariant under τ . If α and $\tau(\alpha)$ are orthogonal to each other, then $\bar{\alpha}$ is a root of medium length in R^{τ} , and otherwise $\bar{\alpha}$ is a short root in R^{τ} . (Remember that the root system BC_n is the union of two root systems of types B_n and C_n such that the long roots of B_n coincide with the short roots of C_n .)

Now let G be of type A_{2n-1} , D_n , or E_6 , and consider the case $\tau^2 = e$. We have seen that τ defines a Lie algebra automorphism via $\tau(X_{\alpha}) = X_{\tau(\alpha)}$ for $\alpha \in \Pi$ and $X_{\alpha} \in L_{\alpha}$. Extending this to the entire Lie algebra, one gets $\tau(X_{\alpha}) = X_{\tau(\alpha)}$ for all $\alpha \in R$. The eigenvectors of $\text{Ad}(\tau)$ are the following: If α is invariant under τ then X_{α} is an eigenvector with eigenvalue 1. If α is not invariant under τ , there are two eigenvectors $X_{\alpha} \pm X_{\tau(\alpha)}$ of eigenvalue ± 1 . Thus for $\tau(\alpha) = \alpha$ and $X = X_{\alpha}$ we have

$$(\text{Ad}(s\tau)^{-1} - I)(X) = (\vartheta_{\bar{\alpha}}(s^{-1}) - 1)X,$$

and

$$(\text{Ad}(s\tau)^{-1} - I)(X) = (\pm \vartheta_{\bar{\alpha}}(s^{-1}) - 1)X$$

for $\tau(\alpha) \neq \alpha$ and $X = X_\alpha \pm X_{\tau(\alpha)}$ respectively. This yields

$$\begin{aligned} & \det(\text{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)}) \\ &= (-2)^{\dim(T/S_0)} \cdot \prod_{\substack{\bar{\alpha} \in R^\tau \\ \tau(\alpha) = \alpha}} (\vartheta_{\bar{\alpha}}(s^{-1}) - 1) \\ & \quad \times \prod_{\substack{\bar{\alpha} \in R^\tau \\ \tau(\alpha) \neq \alpha}} (\vartheta_{\bar{\alpha}}(s^{-1}) - 1)(\vartheta_{\bar{\alpha}}(s^{-1}) - 1) \end{aligned}$$

Multiplying each factor by -1 , using the remark above on the relative length of the $\bar{\alpha}$ as well as the equality

$$(1 - \vartheta_{\bar{\alpha}})(1 + \vartheta_{\bar{\alpha}}) = (1 - \vartheta_{2\bar{\alpha}}),$$

this becomes

$$\begin{aligned} &= (-2)^{\dim(T/S_0)} \cdot \prod_{\substack{\bar{\alpha} \in R^\tau \\ \bar{\alpha} \text{ long}}} (1 - \vartheta_{\bar{\alpha}}(s^{-1})) \prod_{\substack{\bar{\alpha} \in R^\tau \\ \bar{\alpha} \text{ short}}} (1 - \vartheta_{2\bar{\alpha}}(s^{-1})) \\ &= (-2)^{\dim(T/S_0)} \cdot \mathcal{A}(s^{-1}) \bar{\mathcal{A}}(s^{-1}), \end{aligned}$$

with

$$\mathcal{A}(s) = \prod_{\bar{\alpha} \in R_+^{\tau^\vee}} (1 - \vartheta_{\bar{\alpha}}(s)).$$

Here R^{τ^\vee} denotes the dual root system of R^τ which is given by $\bar{\alpha}^\vee = \frac{2\bar{\alpha}}{\langle \bar{\alpha}, \bar{\alpha} \rangle}$ for $\bar{\alpha} \in R^\tau$ and $\langle \cdot, \cdot \rangle$ is a multiple of the Killing form such that $\langle \bar{\alpha}, \bar{\alpha} \rangle = 2$ for a long root $\bar{\alpha} \in R^\tau$.

If G is of type D_4 and $\tau^3 = e$, a similar calculation gives

$$\det(\text{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)}) = 3 \cdot \mathcal{A}(s^{-1}) \bar{\mathcal{A}}(s^{-1}),$$

with $\mathcal{A}(s)$ as above. Observe that in this case R^τ is of type G_2 , so $\dim T/S_0 = 2$.

If G is of type A_{2n} we have to be more careful since R^τ is not reduced and three different root lengths occur. Also, in this case the Lie algebra automorphism τ is slightly more complicated. For $X_\alpha \in L_\alpha$ we have

$$\tau(X_\alpha) = (-1)^{1 + \text{ht}(\alpha)} X_{\tau(\alpha)}.$$

Now $\tau(\alpha) = \alpha$ implies that $\text{ht}(\alpha)$ is even and a similar calculation yields

$$\begin{aligned} & \det(\text{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)}) \\ &= (-2)^{\dim(T/S_0)} \cdot \prod_{\substack{\bar{\alpha} \in R^\tau \\ \bar{\alpha} \text{ long}}} (1 + \vartheta_{\bar{\alpha}}(s^{-1})) \cdot \prod_{\substack{\bar{\alpha} \in R^\tau \\ \bar{\alpha} \text{ middle}}} (1 - \vartheta_{2\bar{\alpha}}(s^{-1})) \\ & \quad \times \prod_{\substack{\bar{\alpha} \in R^\tau \\ \bar{\alpha} \text{ short}}} (1 - \vartheta_{2\bar{\alpha}}(s^{-1})) \end{aligned}$$

But the length of the long roots in BC_n is twice the length of the short roots. So we can put these together to obtain

$$\det(\text{Ad}|_{L(G/S_0)}(s\tau)^{-1} - I|_{L(G/S_0)}) = (-2)^{\dim(T/S_0)} \cdot \Delta(s^{-1}) \bar{\Delta}(s^{-1}),$$

with

$$\Delta(s) = \prod_{\bar{\alpha} \in R_+^1} (1 - \vartheta_{\bar{\alpha}}(s)).$$

Here $R^1 = \{2\bar{\alpha} \mid \bar{\alpha} \in \text{BC}_n, \bar{\alpha} \text{ long}\} \cup \{2\bar{\alpha} \mid \bar{\alpha} \in \text{BC}_n, \bar{\alpha} \text{ middle}\}$. This is a root system of type C_n .

Before stating the integral formula for $G\tau$, we have to compare the different Weyl groups involved. Let T be the maximal torus of G such that $S_0 \subset T$ and let $W(T) = N_G(T)/T$ be the usual Weyl group of G . If we set $W^\tau = \{w \in W(T) \mid \tau w \tau^{-1} = w\}$, then W^τ is the Weyl group of the root system R^τ (and also of course of its dual R^{τ^\vee}).

PROPOSITION 2.4. *Let W^τ be as above. Then*

(i) *There exists a split exact sequence*

$$e \rightarrow (T/S_0)^\tau \rightarrow W(S) \rightarrow W^\tau \rightarrow e.$$

Here $(T/S_0)^\tau$ denotes the fixed point set under conjugation with τ .

(ii) *We have*

$$|W^\tau| = \begin{cases} \frac{1}{2^{\dim(T/S_0)}} \cdot |W(S)| & \text{if } \tau^2 = e \\ \frac{1}{3} \cdot |W(S)| & \text{if } \tau^3 = e \end{cases}$$

Proof. Observe that in our cases we have $W(S) = (N(S) \cap G)/G$ (this is not true for general non-connected Lie groups), so one can define a map $\varphi: W(S) \rightarrow W(T)$, $gS_0 \mapsto gT$. This map is well defined and one has $\text{Im}(\varphi) \subset W^\tau$. The rest is done by a calculation in the Lie algebra of T . For details see [18] (also cf. 3.6). To see that the sequence splits, observe that S_0 is by construction a maximal torus in the connected component of G^τ containing e , and W^τ is the corresponding Weyl group. ■

Putting everything together, one gets

THEOREM 2.5 (“Weyl integral formula” for $G\tau$). *Let $f: G\tau \rightarrow \mathbb{R}$ be a function which is integrable and invariant under conjugation by G . Then*

$$\int_{G\tau} f(g) dg = \frac{1}{|W^\tau|} \int_{S_0} f(s\tau) \Delta(s) \bar{\Delta}(s) ds,$$

with

$$\Delta(s) = \prod_{\bar{\alpha} \in R_+^1} (1 - \vartheta_{\bar{\alpha}}(s)).$$

Here R^1 denotes the root system R^{τ^\vee} if R^τ is reduced and

$$R^1 = \{2\bar{\alpha} \mid \bar{\alpha} \in \text{BC}_n, \bar{\alpha} \text{ long}\} \cup \{2\bar{\alpha} \mid \bar{\alpha} \in \text{BC}_n, \bar{\alpha} \text{ middle}\}$$

is of type C_n if R is of type A_{2n} .

2.3. Applications of the Integral Formula

Let G be a compact connected semisimple Lie group of type A_n , D_n , or E_6 as above, and let $\bar{G} \subset \tilde{G}$ be any non-trivial subextension of G . As a first application of the integral formula we compute the irreducible characters of \bar{G} on the component $G\tau$ for $\tau \in \Gamma$. Let T be the τ -invariant maximal torus of G , and let S be a Cartan subgroup of \bar{G} such that S/S_0 is generated by τS_0 . In the case D_4 , this notation is not quite unique since in general more than one diagram automorphism occurs, but it will always be clear which respective Cartan subgroup is being used at the moment.

Since S_0 is regular in G , we can choose a Weyl chamber $K \subset LT^*$ such that $K \cap LS_0^*$ is not empty and we let \bar{K} denote its closure. Then the set $K^\tau = K \cap LS_0^*$ is a Weyl chamber in LS_0^* with respect to the root system R^τ . Furthermore, let $W = W(T)$ be the Weyl group of G , let I denote the lattice $I = \ker(\exp) \cap LT$, and $I^* \subset LT^*$ its dual.

For linear forms $\lambda \in LT^*$ and $\mu \in LS_0^*$ we define the alternating sums

$$A(\lambda) = \sum_{w \in W(T)} \varepsilon(w) \cdot e(w\lambda)$$

and

$$A^\tau(\mu) = \sum_{w \in W^\tau} \varepsilon(w) \cdot e(w\mu)$$

respectively. Here we have set $\varepsilon(w) = (-1)^{\text{length}(w)}$. Note that for $w \in W^\tau$ the two $\varepsilon(w)$ in the equations above do not necessarily coincide since they come from the presentations of w as an element of two different Weyl groups. In this notation, A^τ is a complex valued function on LS_0 which is alternating with respect to W^τ .

Now let R^1 be the root system used in Theorem 2.5. As usual, we set

$$\rho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha,$$

and

$$\rho^\tau = \frac{1}{2} \sum_{\bar{\alpha} \in R_+^1} \bar{\alpha}.$$

Now we can define functions $\delta: LT \rightarrow \mathbb{C}$, resp. $\delta^\tau: LS_0 \rightarrow \mathbb{C}$ via

$$\delta = e(\rho) \cdot \prod_{\alpha \in R_+} (1 - e(-\alpha))$$

and

$$\delta^\tau = e(\rho^\tau) \cdot \prod_{\bar{\alpha} \in R_+^1} (1 - e(-\bar{\alpha})).$$

The function $\delta^\tau \cdot \bar{\delta}^\tau$ factorizes through \exp , and we have $\delta^\tau \cdot \bar{\delta}^\tau = (\mathcal{A} \cdot \bar{\mathcal{A}}) \circ \exp$, where \mathcal{A} is as in Theorem 2.5. With this notation the classical Weyl character formula for the irreducible character χ_λ of G belonging to the highest weight $\lambda \in I \cap \bar{K}$ reads

$$\chi_\lambda = A(\lambda + \rho) / \delta.$$

Now if $\mu \in I^* \cap LS_0^*$ then $A^\tau(\mu + \rho^\tau) / \delta^\tau$ can be extended uniquely to a function on LS_0 . This function factors through \exp (cf. [2]). In this way $A^\tau(\mu + \rho^\tau) / \delta^\tau$ can be considered as a function on S_0 .

For an arbitrary character χ of \bar{G} we define the function $\chi^\tau: S_0 \rightarrow \mathbb{C}$ via $\chi^\tau(s) = \chi(s\tau)$. So on the connected component $G\tau$ of \bar{G} , the character χ is determined by χ^τ . Now we can state the analogue of the Weyl character formula for the subextensions $\bar{G} \subset \tilde{G}$. First we consider the case when \bar{G} consists of two connected components. So only one diagram automorphism τ is involved, and $\tau^2 = e$.

THEOREM 2.6. *There exists an irreducible character $\tilde{\chi}_\lambda$ of \bar{G} for each $\lambda \in I^* \cap \bar{K}$. If $\lambda \notin LS_0^*$, then*

$$\tilde{\chi}_\lambda|_G = \chi_\lambda + \chi_{\tau(\lambda)},$$

and

$$\tilde{\chi}_\lambda|_{G\tau} \equiv 0.$$

Here χ_λ denotes the irreducible character of G of highest weight λ .

For each $\lambda \in LS_0^*$, there exist two irreducible characters of \bar{G} associated to λ , and we have

$$\tilde{\chi}_\lambda|_G = \chi_\lambda,$$

and

$$\tilde{\chi}_\lambda^\tau = \pm A^\tau(\mu + \rho^\tau)/\delta^\tau.$$

If \bar{G} consists of three connected components, the character formula is essentially the same, except that there are three irreducible characters for each $\lambda \in LS_0^*$. If G is of type D_4 and $\bar{G} = \tilde{G}$, then one can use the character formulas above together with some information about the conjugacy classes in S_3 to determine all irreducible characters. Since this will not be needed in the sequel, we will omit the statement of the result.

Remarks on the Proof. The proof of Theorem 2.6 is essentially the same as the proof of the classical Weyl character formula in [2]. The integral formula is used along with the orthogonality relations for irreducible characters to show that the irreducible characters of \bar{G} must have the given form. Then one can apply the Peter–Weyl theorem to see that each of the functions above must be an irreducible character of \bar{G} . For more details on this see [18].

Remark. Kostant [12] states a character formula for non-connected complex Lie groups which is not quite as explicit as Theorem 2.6. In particular, he gives the character on the component $G\tau$ as a function on $T\tau$, where T is a maximal torus in G . So the different root systems do not appear explicitly (although it is not hard to derive Theorem 2.6 from his formula).

The formula itself was discovered before by Jantzen [8], who calculated the trace of the outer automorphism τ on the weight spaces of an irreducible representation with invariant highest weight of a semisimple algebraic group, and by Fuchs *et al.* [5], who studied the characters of “ τ -twisted” representations of a generalized Kac–Moody algebra.

As in the classical case, there is a Weyl denominator formula:

COROLLARY 2.7. *With the same notation as above we have*

$$\delta^\tau = A^\tau(\rho^\tau).$$

Remark. It is interesting to note that no group belonging to the root system R_1 can, in general, be realized as a subgroup of G . For example, $SO(2n+1)$, or its covering group $Spin(2n+1)$, which are the groups with root system B_n , cannot be realized as subgroups of $SU(2n)$, which belongs to the root system A_{2n-1} .

As a second application of Theorem 2.5 we can derive a formula for the radial part of the Laplacian on $G\tau$ with respect to the G -action by conjugation. The negative of the Killing form on $\mathfrak{g}_\mathbb{C}$ defines a positive definite scalar product on \mathfrak{g} which defines a biinvariant Riemannian metric on G by left translation. We can use right multiplication by τ^{-1} to pull back this Riemannian metric to $G\tau$. So G and $G\tau$ are isomorphic as Riemannian manifolds. This metric is invariant under the G -action on $G\tau$. Let $\Delta_{G\tau}$ be the Laplacian on $G\tau$ with respect to this metric, and let Δ_{S_0} be the Laplacian on S_0 . Now we can use the general theory of radial parts of invariant differential operators [6, Part III, Theorem 3.7] (also cf. Proposition 3.12) to get the following proposition.

PROPOSITION 2.8. *Let $f: G\tau \rightarrow \mathbb{R}$ be a G -invariant function, and $f^\tau: G \rightarrow \mathbb{R}$ be given by $f^\tau(g) = f(g\tau)$. Let δ^τ and ρ^τ be as above. Then we have*

$$\delta^\tau \cdot \Delta_{G\tau}(f) = (\Delta_{S_0} + \|\rho^\tau\|^2)(\delta^\tau \cdot f^\tau),$$

as functions restricted to LS_0 . Here $\|\cdot\|$ is the metric on LS_0^ induced by the negative of the Killing form.*

3. THE AFFINE ADJOINT REPRESENTATION OF A TWISTED LOOP GROUP

3.1. Affine Lie Algebras

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semi simple Lie algebra, with compact involution ω and let \mathfrak{g} be a compact form of $\mathfrak{g}_{\mathbb{C}}$. That is, $i\mathfrak{g} = \{x \in \mathfrak{g}_{\mathbb{C}} \mid \omega(x) = x\}$. The loop algebra $\mathcal{L}\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathcal{L}\mathfrak{g}$) is the algebra of C^∞ -maps from S^1 to $\mathfrak{g}_{\mathbb{C}}$ (resp. \mathfrak{g}). It is a Lie algebra under pointwise Lie bracket. If $\mathfrak{g}_{\mathbb{C}}$ is a complex simple Lie algebra of type X_n and if the circle S^1 is parametrized by the real line \mathbb{R} via the exponential $e(t) = e^{2\pi it}$ then the untwisted affine Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ of type $X_n^{(1)}$ is given by

$$\tilde{\mathfrak{g}}_{\mathbb{C}} = \mathcal{L}\mathfrak{g}_{\mathbb{C}} \oplus \mathbb{C}C \oplus \mathbb{C}D,$$

with Lie bracket

$$\begin{aligned} [x(\cdot) + aC + bD, x_1(\cdot) + a_1C + b_1D] \\ = [x, x_1](\cdot) + bx'_1(\cdot) - b_1x'(\cdot) + (x'(\cdot), x_1(\cdot))C. \end{aligned}$$

Here $x'(t) = \frac{dx(t)}{dt}$, $[x, x_1](t) = [x(t), x_1(t)]$, and

$$(x(\cdot), x_1(\cdot)) = \int_0^1 (x(t), x_1(t)) dt,$$

where (\cdot, \cdot) under the integral sign denotes the Killing form on $\mathfrak{g}_{\mathbb{C}}$.

Remark. We shall adhere to some slight abuse of terminology, here and in the sequel. The affine algebras in the sense of [9] are based on algebraic loops, with finite Fourier expansion. Our algebras may be viewed as completions of these algebraic ones, cf. below. The notation for types will be that of [9].

An invariant bilinear form (\cdot, \cdot) on $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is given by

$$(x(\cdot) + aC + bD, x_1(\cdot) + a_1C + b_1D) = \int_0^1 (x(t), x_1(t)) dt + ab_1 + a_1b.$$

We obtain a so-called compact form $\tilde{\mathfrak{g}}$ of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ by considering $\tilde{\mathfrak{g}} = \mathcal{L}\mathfrak{g} \oplus \mathbb{R}C \oplus \mathbb{R}D$.

If τ is the outer automorphism of $\mathfrak{g}_{\mathbb{C}}$ considered in Section 2.1 and $\text{ord}(\tau) = r$, then the twisted loop algebras $\mathcal{L}(\mathfrak{g}_{\mathbb{C}}, \tau)$ and $\mathcal{L}(\mathfrak{g}, \tau)$ are given by

$$\mathcal{L}(\mathfrak{g}_{(\mathbb{C})}, \tau) = \{x \in \mathcal{L}\mathfrak{g}_{(\mathbb{C})} \mid \tau(x(t)) = x(t + 1/r) \text{ for all } t \in [0, 1]\}.$$

Now if $\mathfrak{g}_{\mathbb{C}}$ is a simple Lie algebra of type X_n with $X_n = A_n, D_n$ or E_6 , then the twisted affine Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ of type $X_n^{(r)}$ is given by

$$\tilde{\mathfrak{g}}_{\mathbb{C}} = \mathcal{L}(\mathfrak{g}_{\mathbb{C}}, \tau) \oplus \mathbb{C}C \oplus \mathbb{C}D$$

with the same Lie bracket as above. The invariant bilinear form (\cdot, \cdot) on the corresponding untwisted affine Lie algebra yields an invariant bilinear form on the twisted affine Lie algebra $\tilde{\mathfrak{g}}_{\mathbb{C}}$ by restriction and it is denoted by the same symbol. The compact form $\tilde{\mathfrak{g}}$ of a twisted affine Lie algebra is obtained in the same way as in the untwisted case.

We now define the C^∞ -topology on the Lie algebras $\tilde{\mathfrak{g}}$. The topology on $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is obtained by viewing it as a direct sum of two copies of $\tilde{\mathfrak{g}}$. If $\mathcal{L}\mathfrak{g}$ is an untwisted loop algebra, one defines the C^∞ -topology on $\mathcal{L}\mathfrak{g}$ via the set of semi-norms

$$p_n(x) = \sup_{\substack{s \leq n \\ t \in [0, 1]}} \left| \frac{d^s x(t)}{dt^s} \right|, \quad n = 1, 2, \dots; x \in \mathcal{L}\mathfrak{g}.$$

With respect to this topology $\mathcal{L}\mathfrak{g}$ is complete. It extends immediately to the enlarged $\tilde{\mathfrak{g}}$ as well as to the twisted subalgebras. We may view these topological algebras as C^∞ -completions of the algebraic loop algebras as, e.g., in [9].

For later applications, we introduce a second topology on the spaces $\mathcal{L}(\mathfrak{g})$ and $\mathcal{L}(\mathfrak{g}, \tau)$. The completions of these spaces with respect to the new topology will not carry a Lie algebra structure any more (cf. [4]). As before, let (\cdot, \cdot) denote the Killing form on $\mathfrak{g}_{\mathbb{C}}$, and let $(\cdot, \cdot)_{\mathfrak{g}}$ denote the negative of the Killing form on \mathfrak{g} . Then $(x, y)_{\mathfrak{g}} = \int_0^1 (x(t), y(t))_{\mathfrak{g}} dt$ gives a scalar product on $\mathcal{L}(\mathfrak{g})$, and by restriction we get a scalar product on $\mathcal{L}(\mathfrak{g}, \tau)$ as well. Let $\mathcal{L}(\mathfrak{g}, \tau)(L_2)$ denote the completion of $\mathcal{L}(\mathfrak{g}, \tau)$ with respect to the metric induced by the scalar product. After extending the scalar product to the completion, $\mathcal{L}(\mathfrak{g}, \tau)(L_2)$ is a Hilbert space.

3.2. Loop Groups and Their Affine Adjoint Representation

From now on, we will write $G_{(\mathbb{C})}$ if we mean either a compact group G or the corresponding complex group $G_{\mathbb{C}}$, and analogously we write $\tilde{\mathfrak{g}}_{(\mathbb{C})}$ for the associated Lie algebras.

If $G_{(\mathbb{C})}$ is a simply connected compact (complex) Lie group then the corresponding untwisted loop group $\mathcal{L}G_{(\mathbb{C})}$ is defined to be the topological group of C^∞ mappings from S^1 to $G_{(\mathbb{C})}$ with pointwise multiplication and

the usual C^∞ -topology. If τ is one of the outer automorphisms of $G_{(\mathbb{C})}$ considered in Section 2.1 and $\text{ord}(\tau)=r$ then the twisted loop group $\mathcal{L}(G_{(\mathbb{C})}, \tau)$ is the subgroup

$$\mathcal{L}(G_{(\mathbb{C})}, \tau) = \{x \in \mathcal{L}G_{(\mathbb{C})} \mid \tau(x(t)) = x(t + 1/r) \text{ for all } t \in [0, 1]\}.$$

Then $\mathcal{L}\mathfrak{g}_{(\mathbb{C})}$ (resp. $\mathcal{L}(\mathfrak{g}_{(\mathbb{C})}, \tau)$) may be viewed as the Lie algebra of $\mathcal{L}G_{(\mathbb{C})}$ (resp. $\mathcal{L}(G_{(\mathbb{C})}, \tau)$) and there are natural adjoint actions of these groups on their Lie algebras (by pointwise finite-dimensional adjoint action). However, for purposes of representation theory it is essential to consider not this action but the adjoint action of the affine Kac–Moody groups on the affine Lie algebras. These groups are given, similar to the Lie algebra case, as central extensions of semidirect products

$$e \rightarrow S^1 \rightarrow \hat{G} \rightarrow \mathcal{L}G \rtimes S^1 \rightarrow e,$$

where the circle in the semidirect product acts on $\mathcal{L}G$ by rotation on the loops (in the complex case the circles S^1 are usually replaced by \mathbb{C}^*), and accordingly in the twisted cases. See [16] for a construction of these groups. Obviously the center of \hat{G} acts trivially in the adjoint representation, thus it is sufficient to consider the action of the quotient group $\mathcal{L}G \rtimes S^1$. It will turn out that the adjoint action of the rotation group S^1 is without relevance for our purposes (i.e, in the case that an $\mathcal{L}G$ -orbit contains a constant loop, see below, this action preserves the $\mathcal{L}G$ -orbit). Therefore, it is sufficient to look at the adjoint action of only the loop group $\mathcal{L}G$ on $\tilde{\mathfrak{g}}$. We call that action, which significantly differs from the usual adjoint action of $\mathcal{L}G$ on its Lie algebra $\mathcal{L}\mathfrak{g}$, the affine adjoint action and denote it by $\tilde{\text{Ad}}$. Exploiting the natural exponential mapping from $\mathcal{L}\mathfrak{g}$ to $\mathcal{L}G$ and working with a fixed faithful matrix representation of G , Frenkel was able to determine the exact form of the affine adjoint action in the untwisted case ([4]).

PROPOSITION 3.1. *Let $g \in \mathcal{L}G_{(\mathbb{C})}$, $y \in \mathcal{L}\mathfrak{g}_{(\mathbb{C})}$ and $a, b \in \mathbb{C}$ (resp. \mathbb{R}), then*

$$\tilde{\text{Ad}} g(aC + bD + y) = \tilde{a}C + bD + gyg^{-1} - bg'g^{-1}$$

with $\tilde{a} = a + (g^{-1}g', y) - \frac{b}{2}(g'g^{-1}, g'g^{-1})$ and $g'(t) = \frac{dg(t)}{dt}$.

The affine adjoint representation of a twisted loop group $\mathcal{L}(G_{(\mathbb{C})}, \tau)$ on the corresponding twisted affine Lie algebra $\tilde{\mathfrak{g}}_{(\mathbb{C})}$ is obtained by restriction of the adjoint representation of $\mathcal{L}G_{(\mathbb{C})}$ on the corresponding untwisted affine Lie algebra $\tilde{\mathfrak{g}}'_{(\mathbb{C})}$ to $\mathcal{L}(G_{(\mathbb{C})}, \tau)$ and $\tilde{\mathfrak{g}}_{(\mathbb{C})}$. Hence the formula of Proposition 3.1 remains true for the twisted cases, as well.

3.3. Classification of Affine Orbits

In the case of an untwisted affine Lie algebra Frenkel and Segal have classified affine adjoint orbits of the loop group $\mathcal{L}G_{(\mathbb{C})}$ on the affine Lie algebra $\tilde{\mathfrak{g}}_{(\mathbb{C})}$ in terms of conjugacy classes of the group $G_{(\mathbb{C})}$ ([4, 16]). By a slight alteration of Frenkel's original methods, we can extend this classification to the twisted loop groups $\mathcal{L}(G_{(\mathbb{C})}, \tau)$. Technically, the study of the affine adjoint action (on elements with $b \neq 0$) is equivalent to that of transformations of ordinary differential equations on the real line with periodic coefficients or, in more advanced terminology, to that of the action of the gauge group on connections in a trivial fibre bundle over the circle S^1 . In the context of twisted loops we shall, in addition, have to work with differential equations with "twisted periodic" coefficients. To obtain a unified statement of the results, we shall allow the twisting automorphism τ to be the identity on $G_{(\mathbb{C})}$. In this case, the group $\mathcal{L}(G_{(\mathbb{C})}, \tau)$ is just the untwisted group $\mathcal{L}G_{(\mathbb{C})}$.

Consider first the system of linear differential equations

$$z'(t) = z(t) x(t),$$

where $x(t), z(t) \in M_n(\mathbb{C})$ for all $t \geq 0$, and $x(t)$ is continuous in t . A fundamental result from the theory of differential equations secures the existence of a unique solution $z(t)$ of the above equation with $z(0) = I_n$. This solution z is usually called the fundamental solution of the differential equation.

Now let x be twisted periodic, that is, $x(t + 1/r) = \tau x \tau^{-1}$ for some invertible matrix τ with $\tau^r = I_n$ and all $t \geq 0$. If z is the fundamental solution of $z' = zx$, then obviously $z_1(t) = \tau^{-1} z(t + 1/r) \tau$ is another solution. Hence there exists a matrix $\tilde{M}(x)$ such that $z_1(t) = \tilde{M}(x) z(t)$. Since we have chosen $z(0) = I_n$, we get $\tilde{M}(x) = z_1(0) = \tau^{-1} z(1/r) \tau$. Now let $M(x) := z(1/r)$ be the " $\frac{1}{r}$ monodromy" of the differential equation $z' = zx$. We then obtain

$$z(t + 1/r) = M(x) \tau z(t) \tau^{-1}$$

for all $t \geq 0$.

For a twisted periodic continuously differentiable g with $g(t) \in Gl_n(\mathbb{C})$ for all $t \geq 0$ let us denote

$$z_g(t) = g(0) z(t) g^{-1}(t)$$

$$x_g(t) = g(t) x(t) g^{-1}(t) - g'(t) g^{-1}(t).$$

Then we have the following proposition.

PROPOSITION 3.2. *Let x be a twisted periodic, continuous, matrix valued function, and let z be the fundamental solution of $z' = zx$. Then*

- (i) $z_g(t)$ is the fundamental solution of $z'_g = z_g x_g$.
- (ii) $M(x_g) = g(0) M(x) \tau g^{-1}(0) \tau^{-1}$.
- (iii) *If x_1 is twisted periodic and there exists a g_0 such that*

$$M(x_1) = g_0 M(x) \tau g_0^{-1} \tau^{-1},$$

then there exists a twisted periodic Matrix $g(t)$ such that $g(t) = g_0$ and $x_g(t) = x_1(t)$ for all $t \geq 0$.

Proof. (i) This is a direct calculation using $(g^{-1})' = -g^{-1}g'g^{-1}$.

(ii) Since g is twisted periodic, we have

$$M(x_g) = z_g(1/r) = g(0) z(1/r) g^{-1}(1/r) = g(0) M(x) \tau g^{-1}(0) \tau^{-1}.$$

(iii) Put $g(t) = z_1^{-1}(t) g_0 z(t)$, where z and z_1 are fundamental solutions of $z' = zx$ and $z'_1 = z_1 x_1$ respectively. Then the same calculation as in the proof of [4, Proposition (3.2.5)] yields $x_g = x_1$. Again, a similar explicit calculation as in [4] gives $g(t + 1/r) = \tau g(t) \tau^{-1}$ for all $t \geq 0$. ■

Before we can use the results above to classify the affine adjoint orbits for arbitrary \mathfrak{g} we need a general fact from differential geometry, which is stated as follows in [4].

PROPOSITION 3.3. *Let $\mathfrak{g}_{\mathbb{C}} \subset M_n(\mathbb{C})$ be a matrix Lie algebra, and $G_{\mathbb{C}} \subset GL_n(\mathbb{C})$ the corresponding Lie group. If z is a solution of the linear differential equation $z' = zx$, then $z(t) \in G_{\mathbb{C}}$ for all $t \geq 0$ if and only if $x(t) \in \mathfrak{g}_{\mathbb{C}}$ for all $t \geq 0$.*

Now let $\tilde{\mathfrak{g}}_{\mathbb{C}}$ be an affine Lie algebra of type $X_n^{(r)}$, and let τ be the corresponding diagram automorphism of the underlying finite dimensional Lie algebra $\mathfrak{g}_{\mathbb{C}}$ used in the loop realization of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. In the case of an untwisted affine Lie algebra, τ is just the identity on $\mathfrak{g}_{\mathbb{C}}$. Let $\tilde{\mathfrak{g}}$ and \mathfrak{g} denote the corresponding compact forms. Following [4], we define an affine shell (“standard paraboloid” in [4]) to be the submanifold of codimension 2 in $\tilde{\mathfrak{g}}_{\mathbb{C}}$

$$\mathcal{P}_{\mathbb{C}}^{a,b} = \{x(\cdot) + a_1 C + b_1 D \in \tilde{\mathfrak{g}}_{\mathbb{C}} \mid 2a_1 b_1 + (x, x) = a, b_1 = b\},$$

where $a, b \in \mathbb{C}$ and $b \neq 0$. The zero-hyperplane in $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is defined to be the subspace

$$\hat{\mathfrak{g}}_{\mathbb{C}} \{x(\cdot) + aC + bD \in \tilde{\mathfrak{g}}_{\mathbb{C}} \mid b = 0\}.$$

By $\mathcal{P}^{a,b}$ with $a, b \in \mathbb{R}$, $a \neq 0$ and \hat{g} we shall denote the corresponding submanifolds of \tilde{g} . Let \mathcal{O}_X be the $\mathcal{L}G_{(\mathbb{C})}$ -Orbit of X in $\tilde{g}_{(\mathbb{C})}$, and let $\mathcal{O}_{g\tau}$ be the $G_{(\mathbb{C})}$ -Orbit of $g\tau$ in $G_{(\mathbb{C})}\tau$. Here $G\tau$ is the connected component of the principal extension \tilde{G} of the compact group G as constructed in Section 2.1, and $\tilde{G}_{\mathbb{C}}$ is the corresponding complexification.

THEOREM 3.4. (i) *Each $\mathcal{L}(G_{\mathbb{C}}, \tau)$ (resp. $\mathcal{L}(G, \tau)$)-orbit in the complex (resp. compact) affine Lie algebra $\tilde{g}_{\mathbb{C}}$ (resp. \tilde{g}) lies either in one of the affine shells $\mathcal{P}_{\mathbb{C}}^{a,b}$ (resp. $\mathcal{P}^{a,b}$) or in the zero-hyperplane.*

(ii) *For a fixed affine shell, the monodromy map*

$$\mathcal{O}_{x(\cdot) + aC + bD} \mapsto \mathcal{O}_{M((1/b)x)\tau}$$

is well defined and injective.

(iii) *For a fixed affine shell, the map defined in (ii) gives a bijection between the $\mathcal{L}(G_{\mathbb{C}}, \tau)$ (resp. $\mathcal{L}(G, \tau)$)-orbit in $\tilde{g}_{\mathbb{C}}$ (resp. \tilde{g}) which contain a constant loop and the $G_{\mathbb{C}}$ (resp. G)-orbit in $G_{\mathbb{C}}\tau$ (resp. $G\tau$) which contain an element that is invariant under conjugation with τ .*

Proof. (i) Follows from the formula in Lemma 3.1. Note that the affine shells in a twisted affine Lie algebra are just the τ -invariant parts of the affine shells in the corresponding untwisted algebras.

(ii) We look at the map $\mathcal{O}_{x(\cdot) + aC + bD} \mapsto \mathcal{O}_{M((1/b)x)\tau}$ where, as above, $M(\frac{1}{b}x) = z(1/r)$, and z is the fundamental solution of $z' = z \cdot \frac{1}{b} \cdot x$. Now, by (3.1), we have

$$\tilde{\text{Ad}}(g)(aC + bD + x) = \tilde{a}C + dD + gxg^{-1} - bg'g^{-1}.$$

But using 3.2(i), we see that $\mathcal{O}_{\tilde{a}C + bD + gxg^{-1} - bg'g^{-1}}$ is being mapped to $\mathcal{O}_{z_g(1/r)\tau}$, and 3.2(ii) yields

$$z_g(1/r) = M\left(\frac{1}{b}x_g\right) = g(0) M\left(\frac{1}{b}x\right) \tau g(0)^{-1} \tau^{-1},$$

hence

$$z_g(1/r) \tau = g(0) z(1/r) \tau g(0)^{-1} \in \mathcal{O}_{z(1/r)\tau}.$$

So the map is well defined and injectivity follows with 3.2(iii).

(iii) If $s\tau \in G_{(\mathbb{C})}\tau$ is invariant under conjugation with τ then so is $r \cdot b \cdot \log(s)$ and $\mathcal{O}_{a_1C + bD + r \cdot b \cdot \log(s)}$ is a preimage of $\mathcal{O}_{s\tau}$ whenever it belongs to $\mathcal{P}^{a,b}$. On the other hand, if the orbit $\mathcal{O}_{a_2C + bD + x(\cdot)}$ contains a constant loop $aC + bD + x_0$, then x_0 has to be invariant under conjugation with τ .

Now the fundamental solution of the differential equation $z' = z \cdot \frac{1}{b}x_0$ is given by $z(t) = \exp(t\frac{1}{b}x_0)$. Hence $z(1/r)$ is invariant under conjugation with τ as well. ■

COROLLARY 3.5. *If $\text{ord}(\tau) = 1$, or G is compact, then the monodromy map in Theorem 3.4 is surjective and hence induces a bijection between the $\mathcal{L}G$ -orbits in a fixed affine shell $\mathcal{P}^{a,b}$ and the $G\tau$ -orbits in $G\tau$.*

Proof. If $\tau = \text{id}$ then the statement is trivial. For compact G we use the fact that every G -orbit in $G\tau$ intersects $S\tau$ in at least one point. Here S is a Cartan subgroup of \tilde{G} containing τ . ■

Remark. In the case of complex groups the classification of $\mathcal{L}(G_{\mathbb{C}}, \tau)$ -orbits in $\tilde{\mathfrak{g}}_{\mathbb{C}}$ remains open. In this case it is no longer true that every $G_{\mathbb{C}}$ -orbit in $G_{\mathbb{C}}\tau$ contains a τ -invariant element (cf. [15] for an example), so different arguments may have to be applied. In any case, it is easy to see that the image under the monodromy map defined in 3.4(ii) are the $G_{(\mathbb{C})}$ -orbit in $G_{(\mathbb{C})}\tau$ for which there exists a C^∞ path $z: [0, 1] \rightarrow G_{(\mathbb{C})}$ such that $z(0) = e$, $z(1/r)\tau \in \mathcal{O}_{g\tau}$ and $z(t + 1/r) = z(1/r)\tau z(t)\tau^{-1}$ for all $t \geq 0$.

Remark. We have not dealt with the $\mathcal{L}(G, \tau)$ -orbits in the zero-hyperplane, which are basically the orbits of the adjoint representation of $\mathcal{L}(G, \tau)$ on its Lie algebra. As we shall see later, the orbits relevant for representation theory are the $\mathcal{L}(G, \tau)$ -orbits in some fixed affine shell with $b \neq 0$. Also, the classification of $\mathcal{L}(G, \tau)$ -orbits in $\hat{\mathfrak{g}}$ is presumably not manageable, i.e. it certainly yields an infinite dimensional “moduli space”.

In a simply connected, semi-simple, compact Lie group, the conjugacy classes are in one-to-one correspondence with a fundamental domain of the affine Weyl group \tilde{W} acting on LT in the notation of Section 2. The affine Weyl group acts like the finite Weyl group of the root system R , belonging to the group G , extended by the group of translations generated by the dual roots $R^* \in LT$. Here, the set of dual roots is given as $R^* = \{\alpha^* \mid \alpha \in R\}$, and $(\alpha^*, \cdot) = \alpha$ where (\cdot, \cdot) is the Killing form. Hence the orbits in a given affine shell $\mathcal{P}^{a,b}$ are in one-to-one correspondence with a fundamental domain of \tilde{W} .

Now, the G -orbits in $G\tau$ are in one-to-one correspondence with the set $S_0\tau/W(S)$, where $W(S) = N(S)/S$ acts by conjugation on $S_0\tau$. We can pull back this action to S_0 by right multiplication with τ^{-1} . Then $tS_0 \in W(S)$ acts on S_0 via $s \mapsto ts\tau t^{-1}\tau^{-1}$. By Proposition 2.4, we have $W(S) = (T/S_0)^\tau \rtimes W^\tau$. In this way, the action of W^τ is the usual Weyl group action of the Weyl group of G^τ on the maximal torus S_0 . Hence the orbits of this action are parametrized by a fundamental domain of the affine Weyl group

\tilde{W}^τ , where the translation part of \tilde{W}^τ is given by the dual roots $R^{\tau*} \in LS_0$ of the root system R^τ belonging to the group G^τ .

An element $tS_0 \in (T/S_0)^\tau$ acts on S_0 via $s \mapsto ts\tau t^{-1}\tau^{-1} = ss_{t^{-1}}\tau$, where $\tau t \tau^{-1} = ts_t$ with $s_t \in S_0$. Viewing LS_0 as the universal covering of S_0 via \exp , we see that $(T/S_0)^\tau$ acts on LS_0 as a group of translations. A direct calculation shows that a set of generators of this group is given by the set

$$\left\{ \frac{1}{r} \sum_{i=1}^r \tau^i(\alpha^*) \mid \alpha^* \in R^*, \tau(\alpha^*) \neq \alpha^* \right\}.$$

So together these two groups yield exactly the action of the affine Weyl group \tilde{W}^1 belonging to the root system R^1 from Section 2. Hence we have proved

PROPOSITION 3.6. *In the twisted compact case, the set of orbits in a given affine shell $\mathcal{P}^{a,b}$ is in one-to-one correspondence with a simplex in LS_0 which is a fundamental domain for the action of the affine Weyl group \tilde{W}^1 belonging to the root system R^1 .*

4. ORBITAL INTEGRALS

4.1. Affine Lie Algebras and the Kac–Weyl Character Formula

Before we start deriving the analogue of Frenkel’s character formula for the twisted affine Lie algebras, let us briefly review some facts from the structure and representation theory of affine Lie algebras. Let $\tilde{\mathfrak{g}}_\mathbb{C}$ be an affine Lie algebra of type $X_n^{(r)}$ with Weyl group \tilde{W} . Let $\tilde{\mathfrak{h}}_\mathbb{C}$ be a Cartan subalgebra and $\Pi = \{\alpha_0, \dots, \alpha_n\} \subset \tilde{\mathfrak{h}}_\mathbb{C}^*$ be a set of simple roots of $\tilde{\mathfrak{g}}_\mathbb{C}$ where the roots are labeled in the usual way (cf. [9]). \tilde{R} denotes the set of roots of $\tilde{\mathfrak{g}}_\mathbb{C}$, and define R° to be the root system which is obtained by deleting the 0th vertex from the extended Dynkin diagram of \tilde{R} . Also, let W° denote the Weyl group belonging to R° . Let $\alpha_i^\vee \in \tilde{\mathfrak{h}}_\mathbb{C}$ be the dual simple roots such that $\langle \alpha_i, \alpha_j^\vee \rangle = (A)_{i,j}$, where A is the generalized Cartan matrix belonging to $\tilde{\mathfrak{g}}_\mathbb{C}$. Let a_i be the “minimal” integers such that $A(a_0, \dots, a_n) = 0$, set $\delta = \sum_{i=0}^n a_i \alpha_i$ and define $d = \frac{1}{2\pi i} D$ with D as in Section 3.1. Then we have $\langle \alpha_i, d \rangle = 0$ and $\langle \delta, d \rangle = 1$. Now, we can define an element $\theta = \delta - a_0 \alpha_0 \in \mathfrak{h}_\mathbb{C}^*$ and the lattice $M = \mathbb{Z} W^\circ \theta^\vee \subset \mathfrak{h}_\mathbb{C}^\circ$. Here $\mathfrak{h}_\mathbb{C}^\circ \subset \tilde{\mathfrak{h}}_\mathbb{C}$ denotes the subspace generated by $\{\alpha_1^\vee, \dots, \alpha_n^\vee\}$, and θ^\vee means the element dual to θ in the sense of [9]. A fundamental result in the theory of affine Lie algebras states that $\tilde{W} = W^\circ M$. Observe that in [9] Kac uses a lattice $M' \subset \mathfrak{h}_\mathbb{C}^*$ after identifying $\mathfrak{h}_\mathbb{C}^*$ with $\mathfrak{h}_\mathbb{C}^\circ$ via an invariant bilinear form.

Turning to the representation theory of affine Lie algebras we define as usual

$$\begin{aligned}\tilde{P} &= \{\lambda \in \tilde{\mathfrak{h}}_{\mathbb{C}}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z} \text{ for all } i=0, \dots, n\}, \\ \tilde{P}_+ &= \{\lambda \in \tilde{P} \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ for all } i=0, \dots, n\}, \quad \text{and} \\ \tilde{P}_{++} &= \{\lambda \in \tilde{P} \mid \langle \lambda, \alpha_i^\vee \rangle > 0 \text{ for all } i=0, \dots, n\}.\end{aligned}$$

Then there exists a bijection between the irreducible integrable highest weight modules of $\tilde{\mathfrak{g}}_{\mathbb{C}}$ and the dominant integral weights $\lambda \in \tilde{P}_+$. If $L(\lambda)$ is an integrable irreducible highest weight module with highest weight $\lambda \in \tilde{P}_+$ then the formal character of $L(\lambda)$ is the formal sum

$$\text{ch } L(\lambda) = \sum_{\lambda \in \tilde{\mathfrak{h}}^*} \dim L(\lambda)_\lambda e(\lambda).$$

Here $L(\lambda)_\lambda$ is the weight space corresponding to the weight λ , and $e(\lambda)$ is a formal exponential. The Kac–Weyl character formula now reads (cf. [9])

$$\text{ch } L(\lambda) = \frac{\sum_{w \in \tilde{W}} \varepsilon(w) e(w(\lambda + \tilde{\rho}))}{e(\tilde{\rho}) \prod_{\alpha \in R} (1 - e(-\alpha))^{\text{mult}(\alpha)}}.$$

Here $\tilde{\rho}$ is defined by $\langle \tilde{\rho}, \alpha_i^\vee \rangle = 1$ for $i=0, \dots, n$ and $\langle \tilde{\rho}, d \rangle = 0$. As usual we have set $\varepsilon(w) = (-1)^{\text{length}(w)}$.

So far, the character was considered as a formal sum involving the formal exponentials $e(\lambda)$. Now we set $e^\lambda(h) = e^{\langle \lambda, h \rangle}$ for $h \in \tilde{\mathfrak{h}} - \mathbb{C}$. In this way one can consider the character of a highest weight $\tilde{\mathfrak{g}}_{\mathbb{C}}$ -module V as an infinite series. Let us set $Y(V) = \{h \in \tilde{\mathfrak{h}}_{\mathbb{C}} \mid \text{ch}_V(h) \text{ converges absolutely}\}$. Then ch_V defines a holomorphic function on $Y(V)$ and the following result holds [9]:

PROPOSITION 4.1. *Let $V(\lambda)$ be the irreducible highest weight module with highest weight $\lambda \in P_+$. Then*

$$Y(V(\lambda)) = \{h \in \tilde{\mathfrak{h}}_{\mathbb{C}} \mid \text{Re}\langle \delta, h \rangle > 0\},$$

where $\delta = \sum_{i=0}^n a_i \alpha_i$ and the a_i are the labels of the vertices of the affine Dynkin diagram (cf. [9]).

In this setting the Kac–Weyl character formula gives an identity of holomorphic functions on $Y(V(\lambda))$.

Now we have $\tilde{\mathfrak{h}}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}}^{\circ} \oplus \mathbb{C}C \oplus \mathbb{C}D$ with C and D as in Section 3.1. With this notation one gets

$$Y(V(\lambda)) = \{h + aC + bD \mid h \in \mathfrak{h}_{\mathbb{C}}^{\circ}, a, b \in \mathbb{C}, \text{Im}(b) < 0\}.$$

4.2. Poisson Transformation of the Numerator of Affine Characters

In this section we will start to derive an analogue of Frenkel's character formula for twisted affine Lie algebras by deriving a formula for the numerator of the character formula in terms of the underlying non-connected Lie group. To do this, we need to introduce some more notation. So let $\tilde{\mathfrak{g}}_{\mathbb{C}}$ be an arbitrary affine Lie algebra of type $X_n^{(r)}$, and let $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{g}}'_{\mathbb{C}}$ be the untwisted affine Lie algebra of type $X_n^{(1)}$ such that $\tilde{\mathfrak{g}}_{\mathbb{C}} \subset \tilde{\mathfrak{g}}'_{\mathbb{C}}$ and let $\tilde{\mathfrak{g}}'$ be the corresponding compact forms. If $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is untwisted, we have $\tilde{\mathfrak{g}}_{\mathbb{C}} = \tilde{\mathfrak{g}}'_{\mathbb{C}}$. Furthermore, let R be the root system of the finite dimensional Lie algebra $\mathfrak{g}_{\mathbb{C}}$ used to construct $\mathcal{L}(\mathfrak{g}_{\mathbb{C}}, \tau)$ in Section 3.1, and for a diagram automorphism τ of $\mathfrak{g}_{\mathbb{C}}$ let R^{τ} be the “folded” root system introduced in Section 2.2. If $\tilde{\mathfrak{g}}_{\mathbb{C}}$ is a twisted affine Lie algebra with root system \tilde{R} , then $R^{\tau} = R^{\circ}$. Also, let (\cdot, \cdot) denote the Killing form on $\tilde{\mathfrak{g}}'_{\mathbb{C}}$, and let $(\cdot, \cdot)_r$ denote its restriction to $\tilde{\mathfrak{g}}_{\mathbb{C}}$.

We now turn to the analytic Kac–Weyl character formula from Section 4.1. Let λ be a highest weight of $\tilde{\mathfrak{g}}_{\mathbb{C}}$. There is no essential loss in generality by assuming $\langle \lambda, d \rangle = 0$. Then, after identifying $\tilde{\mathfrak{h}}_{\mathbb{C}} \cong \tilde{\mathfrak{h}}_{\mathbb{C}}^*$ via $(\cdot, \cdot)_r$, we can choose $a \in \mathbb{C}$ and $H \in \mathfrak{h}_{\mathbb{C}}^{\circ}$ such that $\lambda + \tilde{\rho} = aD + H$. The condition $\lambda \in \tilde{P}_+$ implies $a \in i\mathbb{R}$, $\text{Im } a < 0$ and $H \in i\mathfrak{h}^{\circ}$. The numerator of the Kac–Weyl character formula evaluated at $bD + K$ with $b \in \mathbb{C}$, $\text{Im } b < 0$ and $K \in \mathfrak{h}_{\mathbb{C}}^{\circ}$ now reads

$$\sum_{w \in \tilde{W}} \varepsilon(w) e^{(w(aD + H), bD + K)_r}.$$

Let us assume in the following $b \in i\mathbb{R}$ and $K \in i\mathfrak{h}^{\circ}$. We then set $t = -1/ab$, $h = H/a$ and $k = K/b$ yielding $t \in \mathbb{R}_+$ and $h, k \in \mathfrak{h}$. With $2\pi id = D$, the sum above reads

$$\sum_{w \in \tilde{W}} \varepsilon(w) e^{-(1/t)(w(2\pi id + h), 2\pi id + k)_r}.$$

Let us set $c = 2\pi iC$. Then the lattice M operates on $\tilde{\mathfrak{h}}_{\mathbb{C}}$ via

$$\gamma(h + ac + bd) = h + ac + bd - \left((g, \gamma)_r + \frac{ba_0}{2} \|\gamma\|_r^2 \right) c.$$

Now a short calculation (cf. [4]) shows for $w \in W^{\circ}$, $\gamma \in M$, and $w^{-1}\gamma \in \tilde{W}$

$$(w^{-1}\gamma(2\pi id + h), 2\pi id + k)_r = -\frac{1}{2} \|2\pi ia_0\gamma + h - wk\|_r^2 + \frac{1}{2} \|h\|_r^2 + \frac{1}{2} \|k\|_r^2.$$

Inserting this into the sum above, we get

$$\begin{aligned} & \sum_{w \in \tilde{W}} \varepsilon(w) e^{-(1/t)(w(2\pi id + h), 2\pi id + k)_r} \\ &= e^{-(1/2t) \|h\|_r^2 - (1/2t) \|k\|_r^2} \sum_{\gamma \in 2\pi i a_0 M} \sum_{w \in W^\circ} \varepsilon(w) e^{(1/2t) \|\gamma + h - wk\|_r^2}. \end{aligned}$$

We now need to apply the Poisson transformation formula: For a Euclidean vector space V , a lattice $Q \in V$ and a Schwartz function $f: V \rightarrow \mathbb{C}$ one has

$$\sum_{\mu \in Q^\vee} \hat{f}(\mu) = \text{vol } Q \sum_{\gamma \in Q} f(\gamma)$$

with

$$\hat{f}(\mu) = \int_V e^{2\pi i \langle \gamma, \mu \rangle} f(\gamma) d\gamma.$$

For a fixed $x \in \mathfrak{h}^\circ$ set $f(\mu) = e^{(x, \mu)_r - (t/2) \|\mu\|_r^2}$. Then we get

$$\hat{f}(\gamma) = \left(\frac{2\pi}{t} \right)^{l/2} e^{(1/2t) \|x + 2\pi i \gamma\|_r^2},$$

with $l = \dim_{\mathbb{R}} \mathfrak{h}^\circ$. So for $x = h - wk$ with $h, k \in \mathfrak{h}^\circ$ and $w \in W^\circ$ we obtain the identity

$$\begin{aligned} & \sum_{\gamma \in a_0 M} e^{(1/2t) \|2\pi i \gamma + h - wk\|_r^2} \\ &= \text{vol}(a_0 M)^{-1} \left(\frac{2\pi}{t} \right)^{-l/2} \sum_{\mu \in (a_0 M)^\vee} e^{(\mu, h - wk)_r - (t/2) \|\mu\|_r^2}. \end{aligned}$$

If \tilde{R} is of type Aff1, then θ is a long root in R° , and if \tilde{R} is a root system of type Aff2 or Aff3, but not of type $A_{2n}^{(2)}$, then θ is a short root in R° . In case \tilde{R} is of type $A_{2n}^{(2)}$ then R° is a root system of type BC_n , and θ is a root of medium length in R° .

So if \tilde{R} is of type $X_n^{(r)}$ with $r = 2, 3$ and $\tilde{R} \neq A_{2n}^{(2)}$, then θ^\vee is a long root in $R^{\circ\vee}$, and hence M is the lattice which is generated by the long roots in $R^{\circ\vee}$. If \tilde{R} is of type $A_{2n}^{(2)}$, then θ^\vee is a root of medium length in $R^{\circ\vee}$ and in this case we have $a_0 = 2$. Thus, in all cases, M is the lattice which is generated by the root system R^1 from Section 2.2.

For an arbitrary root system S let $P^\circ(S)$ denote the weight lattice of S . Then the above implies

$$\begin{aligned} & \sum_{\gamma \in a_0 M} e^{(1/2t) \|\mathbf{2\pi i \gamma} + \mathbf{h} - \mathbf{wk}\|_r^2} \\ &= \frac{1}{\text{vol } a_0 M} \left(\frac{2\pi}{t} \right)^{-l/2} \sum_{\mu \in P^\circ(R^1)} e^{(\mu, \mathbf{h} - \mathbf{wk})_r - (t/2) \|\mu\|_r^2}. \end{aligned}$$

Putting the above formulas together, we get

$$\begin{aligned} & \sum_{w \in \tilde{W}} \varepsilon(w) e^{-(1/t)(w(2\pi id + \mathbf{h}), 2\pi id + \mathbf{k})_r} \\ &= \frac{e^{-(1/2t) \|\mathbf{h}\|_r^2 - (1/2t) \|\mathbf{k}\|_r^2}}{\text{vol}(\mathbb{Z}R^1) \left(\frac{2\pi}{t} \right)^{l/2}} \sum_{w \in W^\circ} \sum_{\mu \in P^\circ(R^1)} \varepsilon(w) \\ & \quad \times e^{(\mu, \mathbf{h} - \mathbf{wk})_r - (t/2) \|\mu\|_2^2}. \end{aligned}$$

Now let $W(R^1)$ denote the Weyl group of the root system R^1 . It is a well known fact that (after the choice of a basis of R^1) every weight $\lambda \in P^\circ(R^1)$ is conjugate under $W(R^1)$ to some dominant weight $\lambda' \in P_+^\circ(R^1)$. Since the root systems R^1 and R° are dual to each other, we have $W^\circ = W(R^1)$. So we get

$$\begin{aligned} & \sum_{w \in W^\circ} \sum_{\mu \in P^\circ(R^1)} \varepsilon(w) e^{(\mu, \mathbf{h} - \mathbf{wk})_r - (t/2) \|\mu\|_r^2} \\ &= \sum_{\mu \in P_+^\circ(R^1)} \sum_{w \in W^\circ} \sum_{w' \in W^\circ} \varepsilon(w w') \varepsilon(w') e^{\langle w' \mu, \mathbf{h} - \mathbf{wk} \rangle} e^{-(t/2) \|\mu\|_r^2} \end{aligned}$$

Here we have identified \mathfrak{h}° and $\mathfrak{h}^{\circ*}$ via $(\cdot, \cdot)_r$. In the equation above, the singular weights cancel out, so it is enough to sum over the strictly dominant weights, or equivalently to replace λ by $\lambda + \rho^\tau$ with $\rho^\tau = \frac{1}{2} \sum_{\bar{\alpha} \in R_+^1} \bar{\alpha}$ as in Section 2.2. Hence

$$\begin{aligned} & \sum_{w \in W^\circ} \sum_{\mu \in P^\circ(R^1)} \varepsilon(w) e^{(\mu, \mathbf{h} - \mathbf{wk})_r - (t/2) \|\mu\|_r^2} \\ &= \sum_{\lambda \in P_+^\circ(R^1)} \sum_{w \in W^\circ} \sum_{w' \in W^\circ} \varepsilon(w w') \varepsilon(w') e^{\langle w'(\lambda + \rho^\tau), \mathbf{h} - \mathbf{wk} \rangle} \\ & \quad \times e^{-(t/2) \|\lambda + \rho^\tau\|_r^2} \\ &= \sum_{\lambda \in P_+^\circ(R^1)} \delta^\tau(\mathbf{h}) \delta^\tau(-\mathbf{k}) \chi_\lambda^\tau(\mathbf{h}) \chi_\lambda^\tau(-\mathbf{k}) e^{-(t/2) \|\lambda + \rho^\tau\|_r^2} \end{aligned}$$

with $A^\tau(\lambda)$, δ^τ and $\chi_\lambda^\tau = A^\tau(\lambda + \rho^\tau)/\delta^\tau$ as in Section 2.3. As before, let $\mathfrak{g}_\mathbb{C}$ be the finite dimensional complex Lie algebra used to construct $\mathcal{L}(\mathfrak{g}_\mathbb{C}, \tau)$ with root system R and compact form \mathfrak{g} . Let G be the simply connected compact Lie group belonging to \mathfrak{g} and let $G\tau$ denote the connected component of the non-connected Lie group $G\langle\tau\rangle$ containing τ . In Section 2.3 we have seen that $\chi_\lambda^\tau(h) = \tilde{\chi}_\lambda(e^{h\tau})$ for $h \in \mathfrak{h}^\circ$. Here $\tilde{\chi}_\lambda$ denotes the character of $G \rtimes \langle\tau\rangle$ belonging to the highest weight λ (cf. Theorem 2.6 and observe that in this notation we have $\mathfrak{h}^\circ = LS_0$ with LS_0 as in Section 2.2).

So putting everything together, we have proved the following theorem (which is the analogue of Theorem (4.3.4) in [4]).

THEOREM 4.2. *For $h, k \in \mathfrak{h}^\circ$ one has*

$$\begin{aligned} & e^{(1/2t) \|h\|_r^2} e^{(1/2t) \|k\|_r^2} \sum_{w \in \tilde{W}} \varepsilon(w) e^{-(1/t)(w(2\pi id + h), 2\pi id + k)_r} \\ &= \frac{\delta^\tau(h) \delta^\tau(-k)}{\text{vol}(\mathbb{Z}R^1) \left(\frac{2\pi}{t}\right)^{l/2}} \sum_{\lambda \in P_+^\circ(R^1)} \chi_\lambda(e^{h\tau}) \chi_\lambda(e^{-k\tau}) e^{-(t/2) \|\lambda + \rho^\tau\|_r^2} \end{aligned}$$

Remark. This section is basically a reformulation of the analogous results of [4] in the non-twisted cases and of [11] in the twisted cases. Kleinfeld did his calculations for the twisted affine Lie algebras in concrete realizations of the corresponding root systems. Not realizing the appearance of the characters of the non-connected Lie group $G \rtimes \langle\tau\rangle$, he had to work with the irreducible characters of the different Lie groups corresponding to the root systems R , R^τ and R^{τ^\vee} .

4.3. The Heat Equation

In this section we will see how the expression for the numerator in Theorem 4.2 is connected to the fundamental solution of the heat equation on the component $G\tau$ of the non-connected group $G \rtimes \langle\tau\rangle$. To this end, let Δ_G denote the Laplacian on the compact simply connected group G with respect to the Riemannian metric on G induced by the negative of the Killing form on $\mathfrak{g}_\mathbb{C}$. We can pull back this metric to $G\tau$ such that right multiplication with τ induces an isometry between the Riemannian manifolds G and $G\tau$. The Laplacian on $G\tau$ shall be denoted with $\Delta_{G\tau}$.

Now for a fixed parameter $T > 0$, the heat equation on $G\tau$ reads

$$\frac{\partial f(g\tau, t)}{\partial t} = \frac{sT}{2} \Delta_{G\tau} f(g\tau, t)$$

with $g \in G$, $s \in \mathbb{R}$, $s > 0$ and where $f: G\tau \rightarrow \mathbb{R}$ is continuous in both variables, C^2 in the first and C^1 in the second variable. The fundamental solution of the heat equation is defined by the initial data

$$f(g\tau, t)|_{t=+0} = \delta_\tau(g\tau),$$

where δ_τ is the Dirac delta distribution centered at $\tau \in G\tau$.

For a highest weight $\lambda \in P_+^\circ(R)$ of G let $d(\lambda)$ denote the dimension of the corresponding irreducible representation of G and χ_λ its character. Then the fundamental solution of the heat equation on G is given by

$$u_s(g, t) = \sum_{\lambda \in P_+^\circ(R)} d(\lambda) \chi_\lambda(g) e^{-(stT/2)(\|\lambda + \rho\|^2 + \|\rho\|^2)}$$

(see [3]). Now G and $G\tau$ are isometric as Riemannian manifolds, hence the fundamental solutions of the respective heat equations coincide. That is, the fundamental solution of the heat equation on $G\tau$ is given by

$$v_s(g\tau, t) = \sum_{\lambda \in P_+^\circ(R)} d(\lambda) \chi_\lambda(g) e^{-(stT/2)(\|\lambda + \rho\|^2 - \|\rho\|^2)}.$$

There is a well known identity for the characters of a compact group which can be derived using the orthogonality relations for irreducible characters:

$$d(\lambda) \int_G \chi_\lambda(g_1 g g_2^{-1} g^{-1}) dg = \chi_\lambda(g_1) \chi_\lambda(g_2^{-1}),$$

where dg denotes the normalized Haar measure on G . Using a version of the orthogonality relations for non-connected groups, it is easy to prove an analogous formula for the characters on the outer components:

$$d(\lambda) \int_G \chi_\lambda(g_1 \tau g \tau^{-1} g_2^{-1} g^{-1}) dg = \chi_\lambda(g_1 \tau) \chi_\lambda(\tau^{-1} g_2^{-1}).$$

Hence we obtain

$$\begin{aligned} & \int_G v_s(gg_1 \tau g^{-1} \tau^{-1} g_2^{-1} \tau, t) dg \\ &= \sum_{\lambda \in P_+^\circ(R)} \chi_\lambda(g_1 \tau) \chi_\lambda(\tau^{-1} g_2^{-1}) e^{-(stT/2)(\|\lambda + \rho\|^2 - \|\rho\|^2)}. \end{aligned}$$

By Theorem 2.6 we see that $\chi_\lambda(g\tau) = 0$ if λ is not τ -invariant. Furthermore, we have $\|\lambda + \rho\| = \|\lambda + \rho^\tau\|_r^2$ if λ is τ -invariant. Thus, using Theorem 4.2,

exchanging the role of s and t , and fixing the parameter value $s = T$, we have proved the following proposition.

PROPOSITION 4.3. *For $h, k \in \mathfrak{h}^\circ$ one has*

$$\begin{aligned} & \sum_{w \in \tilde{W}} \varepsilon(w) e^{-(1/t)(w(2\pi id + h), 2\pi id + k)_r} \\ &= \frac{e^{-(1/2t) \|h\|_r^2} e^{-(1/2t) \|k\|_r^2} e^{-(t/2) \|\rho^\tau\|_r^2} \delta^\tau(h) \delta^\tau(-k)}{\text{vol}(\mathbb{Z}R^1) \left(\frac{2\pi}{t}\right)^{l/2}} \\ & \quad \times \int_V v_{t/T^2}(ge^h \tau g^{-1} \tau^{-1} e^{-k} \tau, T) dg \end{aligned}$$

4.4. Wiener Measures and a Path Integral

The main result of this subsection will be a further reformulation of the numerator of the character formula as an integral over a certain path space on the connected component $G\tau$. This is based on Proposition 4.3, above, and the theory of Wiener measure on $G\tau$ which we will study first. (Compare with [4] and consult, e.g., [13] for a comprehensive treatment of the theory of Wiener measures on a vectorspace.)

The Wiener measure on an Euclidean vectorspace V of variance $s > 0$ is a measure ω_V^s on the Banach space of paths

$$C_V = \{x: [0, T] \rightarrow V \mid x(0) = 0, x \text{ continuous}\}$$

(with the supremum norm) and is defined using the fundamental solution $w_s(x, t)$ of the heat equation

$$\frac{\partial f(x, t)}{\partial t} = \frac{sT}{2} \Delta_V f(x, t).$$

on V as follows: First, one defines cylinder sets in C_V to be the subsets of C_V

$$\{x \in C_V: (x(t_1) \in A_1, \dots, x(t_m) \in A_m)\},$$

with $0 < t_1 \leq t_2 \leq \dots \leq t_m \leq T$, $m \in \mathbb{N}$, and where A_1, \dots, A_m are Borel sets in V . Then the Wiener measure ω_V^s of variance $s > 0$ is defined on the cylinder sets of C_V via

$$\begin{aligned} & \omega_V^s(x(t_1) \in A_1, \dots, x(t_m) \in A_m) \\ &= \int_{A_1} \dots \int_{A_m} w_s(\Delta x_1, \Delta t_1) \dots w_s(\Delta x_m, \Delta t_m) dx_1 \dots dx_m, \end{aligned}$$

where dx is a Lebesgue measure on V and we have set $x_k = x(t_k)$, $\Delta x_k = x_k - x_{k-1}$, $\Delta t_k = t_k - t_{k-1}$ and $x_0 = 0$.

The conditional Wiener measure $\omega_{V,Z}^s$ of variance $s > 0$ is defined on the closed subspace $C_{V,X} \subset C_V$ with fixed endpoint $x(T) = X$ on the cylinder sets via

$$\begin{aligned} \omega_{V,X}^s(x(t_1) \in A_1, \dots, x(t_{m-1}) \in A_{m-1}) \\ = \int_{A_1} \cdots \int_{A_{m-1}} w_s(\Delta x t_1, \Delta t_1) \cdots w_s(\Delta x_m, \Delta t_m) dx_1 \cdots dx_{m-1}, \end{aligned}$$

where additionally $x_m = X$ and $t_m = T$.

Now a classical result in the theory of Wiener measures states that the measures ω_V^s and $\omega_{V,X}^s$ are σ -additive on the σ -field generated by the cylinder sets in C_V and $C_{V,X}$ respectively. Furthermore, the σ -fields generated by the cylinder sets are exactly the Borel fields of the respective Banach spaces (cf. [13]). As another result, we have

$$\omega_V^s(C_V) = 1$$

and

$$\omega_{V,X}^s(C_{V,X}) = w_s(X, T),$$

where $w_s(x, t)$ is the fundamental solution of the heat equation on V .

Using the fundamental solution $u_s(g, t)$ of the heat equation on the compact Lie group G , one can define the Wiener measure ω_G^s and the conditional Wiener measure $\omega_{G,Z}^s$ on the complete metric spaces

$$C_G = \{z: [0, T] \rightarrow G \mid z(0) = e, z \text{ continuous}\}$$

and $C_{G,Z} = \{z \in C_G, z(T) = Z\}$ in exactly the same fashion as the Wiener measure on the vectorspace V (cf. [4]). The metric ϱ on C_G is given by $\varrho(z, z_1) = \sup_{t \in [0, T]} \varrho_0(z(t), z_1(t))$, where $\varrho_0(g, g_1)$ denotes the length of a shortest geodesic in G connecting two given points g and g_1 (the metric on G still being given by the negative of the Killing form on \mathfrak{g}).

There is an important connection between the Wiener measure on G and the Wiener measure on the Lie algebra \mathfrak{g} which was discovered by Ito [7] and explicitly constructed by McKean [14]: Let $y \in C_{\mathfrak{g}}$ be a continuous path. Then for $n \in \mathbb{N}$ and $k = 0, \dots, 2^n - 1$, we define a path $z_n: [0, T] \rightarrow G$ by $z_n(0) = e$ and

$$z_n(t) = z_n\left(\frac{k}{2^n} T\right) \exp\left(y(t) - y\left(\frac{k}{2^n} T\right)\right) \quad \text{for } \frac{k}{2^n} T < t \leq \frac{k+1}{2^n} T.$$

Note that if y is a differentiable path, then $\lim_{n \rightarrow \infty} z_n$ is the fundamental solution of the differential equation of $z' = zy'$ and hence a well defined path in G . Let us define a map

$$i: G_{\mathfrak{g}} \rightarrow C_G$$

via

$$y \mapsto \begin{cases} \lim_{n \rightarrow \infty} z_n & \text{if the limits exists,} \\ e & \text{else,} \end{cases}$$

The fundamental result of Ito and McKean states that the series z_n , converges with $n \rightarrow \infty$ in the topology of C_G almost everywhere with respect to the measure $\omega_{\mathfrak{g}}^s$. Furthermore, the measure on C_G induced by the map i coincides with the Wiener measure ω_G^s on G . Hence i induces an isomorphism

$$I: L_1(C_{\mathfrak{g}}, \omega_{\mathfrak{g}}^s) \rightarrow L_1(C_G, \omega_G^s)$$

via $If(iy) = f(y)$.

This isomorphism is called Ito's isomorphism in the literature. With its help it is easy to translate most of the results about the Wiener measure on a vectorspace to a corresponding result about Wiener measure on a compact Lie group. For example we have

$$\omega_G^s(C_G) = 1$$

and

$$\omega_{G,Z}^s(C_G, Z) = u_s(Z, T),$$

where $u_s(g, t)$ denotes the fundamental solution of the heat equation on G (cf. [4]).

We will denote the integrals with respect to the Wiener measure and the conditional Wiener measure on V as

$$\int_{C_V} f(x) d\omega_V^s(x),$$

resp.

$$\int_{C_{V,X}} f(x) d\omega_{V,X}^s(x),$$

and accordingly for the Wiener measures on G .

One of the most important properties of these integrals is its translation quasi-invariance (cf. [13]): Let $f: C_V \rightarrow \mathbb{R}$ be an integrable function and let $y \in C_V$ be a C^∞ -path. Then

$$\int_{C_V} f(x) d\omega_V^s(x) = \int_{C_V} f(x+y) e^{-(1/s)(x', y') - (1/2s)(y', y')} d\omega_V^s(x).$$

For a function $f: C_{V, X} \rightarrow \mathbb{R}$ the translation quasi-invariance of the conditional Wiener measure reads

$$\begin{aligned} \int_{C_{V, X}} f(x) d\omega_{V, X}^s(x) \\ = \int_{C_{V, X+Y}} f(x+y) e^{-(1/s)(x', y') - (1/2s)(y', y')} d\omega_{V, W+Y}^s(x), \end{aligned}$$

with $Y = y(T)$. In the above formulas, (x', y') denotes the Stieltjes integral $\frac{1}{T} \int_0^T (y'(t), dx(t))$, and (\cdot, \cdot) denotes the scalar product on V .

Translated to the Wiener integral on G , the translation quasi-invariance looks as follows (see [4]):

PROPOSITION 4.4. (i) *Let $f: C_G \rightarrow \mathbb{R}$ be an integrable function, and $g \in C_G$ be a C^∞ -path. Then*

$$\begin{aligned} \int_{C_G} f(z) d\omega_G^s(z) \\ = \int_{C_G} f(zg) e^{-(1/s)(z^{-1}z', g'g^{-1})_{\mathfrak{g}} - (1/2s)(g^{-1}g', g^{-1}g')_{\mathfrak{g}}} d\omega_G^s(z). \end{aligned}$$

(ii) *Let $f: C_{G, Z} \rightarrow \mathbb{R}$ be an integrable function, and $g \in C_G$ be a C^∞ -path. Then*

$$\begin{aligned} \int_{C_{G, Z}} f(z) d\omega_{G, Z}^s(z) = \int_{C_{G, Zg(T)^{-1}}} f(zg) e^{-(1/s)(z^{-1}z', g'g^{-1})_{\mathfrak{g}}} \\ \times e^{-(1/2s)(g^{-1}g', g^{-1}g')_{\mathfrak{g}}} d\omega_{G, Zg(T)^{-1}}^s(z). \end{aligned}$$

Here the term $(z^{-1}z', g'g^{-1})_{\mathfrak{g}}$ should be interpreted as the Stieltjes integral $\frac{1}{T} \int_0^T (g'g^{-1}, d(i^{-1}(z)))_{\mathfrak{g}}$, where $(\cdot, \cdot)_{\mathfrak{g}}$ denotes the negative of the Killing form on \mathfrak{g} (we add the subscript \mathfrak{g} here to avoid possible confusions in later calculations). Note that i^{-1} is a well defined map almost everywhere on C_G with respect to ω_G^s .

Using the translation quasi-invariance of the Wiener measure on G , Frenkel computes the following integral with respect to this measure [4, Proposition 5.2.12]:

LEMMA 4.5. *Let $Y \in \mathcal{L}(\mathfrak{g}, \tau)$ and let $g \in C_G$ be a C^∞ -path such that $g' = gY$. Then*

$$e^{-\|Y\|_{\mathfrak{g}}^2/2s} \int_{C_{G,Z}} e^{(1/s)(z^{-1}z', Y)_{\mathfrak{g}}} d\omega_{G,Z}^s(z) = u_s(Zg(T)^{-1}, T),$$

where $u_s(z, t)$ is the fundamental solution of the heat equation on G .

Now let $\mathcal{O}_{g\tau}$ denote the G -orbit in $G\tau$ containing the element $g\tau$ (cf. Section 3.3). Multiplying each element of $G\tau$ with τ^{-1} , we can identify $\mathcal{O}_{g\tau}$ with a G -orbit in G , where G acts on itself by twisted conjugation: $(h, g) \mapsto hg\tau h^{-1}\tau^{-1}$. This orbit will be denoted with $\mathcal{O}_{g\tau}$ as well. We can now define $C_{G, \mathcal{O}_{g\tau}} \subset C_G$ to be the space of continuous paths with $z: [0, T] \rightarrow G$ with $z(T) \in \mathcal{O}_{g\tau}$. A conditional Wiener measure $\omega_{G, \mathcal{O}_{g\tau}}^s$ on $C_{G, \mathcal{O}_{g\tau}}$ is defined via

$$\begin{aligned} & \int_{C_{G, \mathcal{O}_{g\tau}}} f(z) d\omega_{G, \mathcal{O}_{g\tau}}^s(z) \\ &= \int_G \left(\int_{C_{G, g_1 g\tau g_1^{-1}\tau^{-1}}} f(z) d\omega_{G, g_1 g\tau g_1^{-1}\tau^{-1}}^s(z) \right) dg_1, \end{aligned}$$

where f is integrable on $C_{G, g_1 g\tau g_1^{-1}\tau^{-1}}$ for almost all $g_1 \in G$. Inserting this definition into Lemma 4.5, one gets:

COROLLARY 4.6. *Let $Y \in \mathcal{L}(\mathfrak{g}, \tau)$ and let $g \in C_G$ be a C^∞ -path such that $g' = gY$. Then*

$$\begin{aligned} & e^{-\|Y\|_{\mathfrak{g}}^2/2s} \int_{C_{G, \mathcal{O}_{Z\tau}}} e^{(1/s)(z^{-1}z', Y)_{\mathfrak{g}}} d\omega_{G, \mathcal{O}_{Z\tau}}^s(z) \\ &= \int_G u_s(g_1 Z\tau g_1^{-1}\tau^{-1}g(T)^{-1}, T) dg_1, \end{aligned}$$

where $u_s(z, t)$ is the fundamental solution of the heat equation on G .

We want to interpret the numerator of the Kac–Weyl character formula as an integral in a space of paths in $G\tau$, so we need to define a Wiener measure on the space

$$C_{G\tau} = \{ \tilde{z}: [0, T] \rightarrow G\tau \mid \tilde{z}\tau^{-1} \in C_G \}.$$

This can be done with the help of the fundamental solution $v_s(g\tau, t)$ on $G\tau$ in exactly the same way as on G and on V . But as we have seen before, we have $v_s(g\tau, t) = u_s(g, t)$ for all $g \in G$ and $t > 0$. So the integrals on G and $G\tau$ will not differ, and we can define the measure $\omega_{G\tau}^s$ directly by

$$\int_{C_{G\tau}} f(\tilde{z}) d\omega_{G\tau}^s(\tilde{z}) = \int_{C_G} \tilde{f}(\tilde{z}\tau^{-1}) d\omega_G^s(\tilde{z}\tau^{-1}),$$

where \tilde{f} is a function on C_G which is given by $\tilde{f}(z) = f(z\tau)$. The conditional Wiener measures on $C_{G\tau, Z\tau}$ and $C_{G\tau, \mathcal{O}_{G\tau}}$ are defined analogously.

So the formula in Corollary 4.6 now reads

$$\begin{aligned} e^{-\|Y\|_{\mathfrak{g}}^2 T^2/2t} \int_{C_{G\tau, \mathcal{O}_{Z\tau}}} e^{(T^2/t)(z^{-1}z', Y)_{\mathfrak{g}}} d\omega_{G\tau, \mathcal{O}_{Z\tau}}^{t/T^2}(z\tau) \\ = \int_G v_{t/T^2}(g_1 Z\tau g_1^{-1} \tau^{-1} g(T)^{-1} \tau, T) dg_1, \end{aligned}$$

where $v_s(g\tau, t)$ is the fundamental solution of the heat equation on $G\tau$. Note, that we also have set $s = \frac{1}{T^2}$ in the above calculation.

Now we fix the parameter value $T = \frac{1}{r}$. Observe that for $Y \in \mathcal{L}(\mathfrak{g}, \tau)$ we then have $\|Y\|_r = -\|Y\|_{\mathfrak{g}}$. So for $h, k \in \mathfrak{h}^\circ$, we can set $Y = \frac{1}{T}k = rk$ and $Z = e^h$. Then we have $g(T)^{-1} = e^{-k}$. Hence the integral formula above and Proposition 4.3 yield:

PROPOSITION 4.7. *Let $h, k \in \mathfrak{h}^\circ$. Then*

$$\begin{aligned} \sum_{w \in \tilde{W}} \varepsilon(w) e^{-(1/t)(w(2\pi id + h), 2\pi id + k)_r} \\ = \frac{\delta^\tau(h) \delta^\tau(-k) e^{-(1/2t)\|h\|_r - t/2\|\rho^\tau\|_r}}{\text{vol}(\mathbb{Z}R^1) \left(\frac{2\pi}{t}\right)^{l/2}} \\ \times \int_{C_{G\tau, \mathcal{O}_{e^h\tau}}} e^{(1/tr^2)(z^{-1}z', k)_{\mathfrak{g}}} d\omega_{G\tau, \mathcal{O}_{e^h\tau}}^{tr^2}(z\tau). \end{aligned}$$

4.5. Affine Characters and Orbital Integrals

In this section we will indicate, how the integral in Proposition 4.7 can be interpreted as an integral over an affine coadjoint orbit of $\mathcal{L}(G, \tau)$. This interpretation and the analytic Kac–Weyl character formula from Section 4.1 will then yield an analogue of the Kirillov character formula for compact semisimple Lie groups. For precise details we have to refer to Frenkel’s work [4].

For fixed $a, b \in \mathbb{R}$ and $b \neq 0$ let $\mathcal{P}^{a,b}$ be the affine shell defined in Section 3.3. We can identify $\mathcal{P}^{a,b}$ with $\mathcal{L}(\mathfrak{g}, \tau)$ via the projection $p: C \mapsto 0$ and $D \mapsto 0$. Under this projection, the affine adjoint $\mathcal{L}(G, \tau)$ -action on $\mathcal{L}(\mathfrak{g}, \tau)$ is given by $(g, y) \mapsto gyg^{-1} - bg'g^{-1}$ (cf. Proposition 3.1).

We have a series of maps

$$\mathcal{P}^{a,b} \xrightarrow{p} \mathcal{L}(\mathfrak{g}, \tau) \xrightarrow{s} C_{\mathfrak{g}}^{\infty} \xrightarrow{i} C_G^{\infty} \xrightarrow{e_{\tau}} G_{\tau},$$

with $s(x)(t) = \int_0^t x(\kappa) d\kappa$, and where i maps a path $y \in C_{\mathfrak{g}}^{\infty}$ to the fundamental solution of the differential equation $z' = \frac{1}{b}zy'$. The map e_{τ} is given by $e_{\tau}(z) = z(1/r)\tau$. From Ito's isomorphism we have a map $\tilde{i}: C_{\mathfrak{g}} \rightarrow C_G$, which is the extension of the map $i: C_{\mathfrak{g}}^{\infty} \rightarrow C_G^{\infty}$ above to the corresponding completions $C_{\mathfrak{g}}$ and C_G .

Now every element $y \in C_{\mathfrak{g}}$ defines an element $dy \in \mathcal{L}(\mathfrak{g}, \tau)^*$ via the Stieltjes integral

$$\langle x, dy \rangle = r \int_0^{1/r} (x(\kappa), dy(\kappa)),$$

where (\cdot, \cdot) denotes the Killing form on \mathfrak{g} as in Sections 3.1 and 4.1. Note that for $y \in \mathcal{L}(\mathfrak{g}, \tau)$, we have $\langle x, d(s(y)) \rangle = (x, y)_r$, where $(\cdot, \cdot)_r$ is the bilinear form on $\mathcal{L}(\mathfrak{g}, \tau)$ defined in Section 4.1. Let $\mathcal{L}(\mathfrak{g}, \tau)_0^*$ denote the image of $C_{\mathfrak{g}}$ under this map with the topology induced from $C_{\mathfrak{g}}$, and let $\tilde{s}: \mathcal{L}(\mathfrak{g}, \tau)_0^* \rightarrow C_{\mathfrak{g}}$ denote the inverse map. In this notation, \tilde{s} is the extension of the map s to $\mathcal{L}(\mathfrak{g}, \tau)_0^*$. Putting the above remarks together, we get the following commutative diagram:

$$\begin{array}{ccccccc} \mathcal{P}^{a,b} & \xrightarrow{p} & \mathcal{L}(\mathfrak{g}, \tau) & \xrightarrow{s} & C_{\mathfrak{g}}^{\infty} & \xrightarrow{i} & C_G^{\infty} \xrightarrow{e_{\tau}} G_{\tau} \\ & & \cap & & \cap & & \cap \\ & & \mathcal{L}(\mathfrak{g}, \tau)_0^* & \xrightarrow{\tilde{s}} & C_{\mathfrak{g}} & \xrightarrow{\tilde{i}} & C_G \xrightarrow{\tilde{e}_{\tau}} G_{\tau} \end{array}$$

Here \tilde{e}_{τ} denotes the extension of e_{τ} to C_G . We have seen in Section 3.3 that under the composition $e_{\tau} \circ i \circ s \circ p$, a $\mathcal{L}(G, \tau)$ -orbit $\mathcal{O}_{x(\cdot) + a_1 C + b_1 D} \subset \mathcal{P}^{a,b}$ is mapped to the G -orbit $\mathcal{O}_{e_{\tau} \circ i \circ s(x)} \subset G_{\tau}$.

On the Hilbert space $\mathcal{L}(\mathfrak{g}, \tau)(L_2)$ introduced Section 3.1, we can define a norm by

$$|x(\cdot)| = \sup_{t \in [0, 1/r]} \left| \int_0^t x(\kappa) d\kappa \right|.$$

The completion of $\mathcal{L}(\mathfrak{g}, \tau)(L_2)$ with respect to this norm will be $\mathcal{L}(\mathfrak{g}, \tau)_0^*$. So we have a series of completions

$$\mathcal{L}(\mathfrak{g}, \tau) \subset \mathcal{L}(\mathfrak{g}, \tau)(L_2) \subset \mathcal{L}(\mathfrak{g}, \tau)_0^*,$$

with respect to the L_2 -topology on $\mathcal{L}(\mathfrak{g}, \tau)$ and the norm on $\mathcal{L}(\mathfrak{g}, \tau)(L_2)$ introduced above.

In this picture, the set $\tilde{s}^{-1} \circ \tilde{t}^{-1} \circ \tilde{e}_\tau^{-1}(\mathcal{O}_{e_\tau \circ i \circ s(x)}) \subset \mathcal{L}(\mathfrak{g}, \tau)_0^*$ can be viewed as the closure of the affine adjoint orbit $\mathcal{O}_{x(\cdot) + a_1 C + b_1 D}$ in $\mathcal{L}(\mathfrak{g}, \tau)_0^*$ and is mapped to $C_{G, \mathcal{O}_{i \circ s(x)}}$ under the map $\tilde{t} \circ \tilde{s}$. Accordingly, the integral

$$\int_{C_{G\tau, \mathcal{O}_{Z\tau}}} f(z) d\omega_{G\tau, \mathcal{O}_{Z\tau}}^s(z\tau)$$

can be viewed as an integral over the closure of the corresponding affine adjoint orbit in $\mathcal{L}(\mathfrak{g}, \tau)_0^*$. For more details, which involve the construction of a Gaussian measure on $\mathcal{L}(\mathfrak{g}, \tau)_0^*$, cf. [4].

The discussion above allows us to interpret the integral appearing in the formula for the numerator of the Kac–Weyl character formula in Proposition 4.7 as an integral over the closure in $\mathcal{L}(\mathfrak{g}, \tau)_0^*$ of the affine adjoint orbit containing $aD + H = \Lambda + \tilde{\rho}$. The denominator of the Kac–Weyl character formula is a function p not depending on the highest weight Λ of the corresponding representation and hence can be seen as the analogue of the universal function appearing in the Kirillov character formula for compact Lie groups (in the context of compact Lie groups, the universal function is given by the denominator of the Weyl character formula as well). Hence our character formula for affine Lie algebras can be written in the following way:

THEOREM 4.8. *Let $bD + K \in \tilde{h}$ with $K \in \mathfrak{h}^\circ$ and $b \in i\mathbb{R}$, $\text{Im}(b) < 0$. Furthermore, for $\Lambda \in \tilde{P}_+$ let $\Lambda + \tilde{\rho} = aD + H$. Then the character of the highest weight representation corresponding to Λ evaluated at $bD + K$ is given by*

$$\text{ch } L(\Lambda)(bD + K)$$

$$= p^{-1}(bD + K) \cdot \frac{\delta^\tau \left(\frac{H}{a} \right) \delta^\tau \left(-\frac{K}{b} \right) e^{ab/2 \|H/a\|_r - 1/2ab \|\rho^\tau\|_r}}{\text{vol}(\mathbb{Z}R^1)(-2ab\pi)^{1/2}}$$

$$\times \int_{C_{g\tau, \mathcal{O}_{(eH/a)\tau}}} e^{-(ab/r^2)(z^{-1}z', K/b)_\mathfrak{g}} d\omega_{G\tau, \mathcal{O}_{(eH/a)\tau}}^{-r^2/ab}(z\tau).$$

Following [4], the path integral above can be interpreted as a Gaussian integral over the closure of the affine orbit containing $aD + H$ in $\mathcal{L}(\mathfrak{g}, \tau)_0^*$. In that setup, the above formula may be seen as an exact analogue of Kirillov's classical character formula.

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